GRADED ENDOMORPHISM RINGS AND EQUIVALENCES

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ABSTRACT. Let R be a G-graded ring, M a G-graded Σ-quasiprojective R-module, and $E = \mathrm{END}_R(M)^{\mathrm{op}}$ its graded ring of endomorphisms. For any subgroup H of G, we prove that certain full subcategories of G/H-graded R-modules associated with M are equivalent to a quotient category of G/H-graded E-modules determined by the idempotent G-graded ideal of E consisting of endomorphisms which factor through a finitely generated submodule of M. Properties and applications of these equivalences are also examined.

INTRODUCTION

Let R be a ring, M be a left R-module and $S = \mathrm{End}_R(M)^{\mathrm{op}}$. Assuming that M is Σ-quasiprojective, J.L. García Hernández and J.L. Gómez Pardo proved in [7, Theorem 1.3] that the functors $\mathrm{Hom}_R(M)$ and $M \otimes_S -$ induce an equivalence between the full subcategory of M-presented R-module and a certain subcategory of S-Mod, which coincide with S-Mod if and only if M is finitely generated. Two other realization theorems as subcategories of R-Mod of the same quotient category of S-Mod were given by the same authors, and the equivalences were generalized in [8] in the context of Grothendieck categories.

Now assume that $R = \bigoplus_{g \in G} R_g$ and $M = \bigoplus_{g \in G} M_g$ are graded by a group G. Then S has a subring $E = \mathrm{END}(M)^{\mathrm{op}}$ which is also G-graded, and $E = S$ if G is finite or M is finitely generated. If M is finitely generated, it was shown in [11] that the above equivalence preserves the modules graded by (transitive) G-sets, and it is compatible whith the grade forgetting functor. This kind of graded equivalences has appeared in two contexts. One of them is the Clifford theory of graded rings, as developed especially by E. Dade in [5], where M is assumed to be a simple object of the category R-Gr of G-graded R-modules. The other is when M is taken to be the canonical generator

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$\bigoplus_{g \in G} R(g)$ of R-Gr. Several authors have studied the case of infinite G, and the results of T. Albu and C. Năstăsescu [3] are among our main starting points here.

The aim of the present paper is to establish equivalences between categories of modules graded by G-sets, which cover and unify the above results. The main difficulty is that we want to have graded modules in both sides of the equivalence, so we have to deal with the graded ring E instead of S, and the functor $\text{Hom}_R(M, -)$ must be replaced as well. The paper is organized as follows. In Section 1 we provide the necessary background material on torsion theory and on graded rings and modules. In Section 2 we develop to some extent a theory of rigid closed and localizing subcategories of modules graded by G-sets, and their Gabriel topology counterpart. We are concerned with the behaviour of the grade forgetting functor and its right adjoint with respect to these subcategories. Such a study was initiated in [10] and [16] in the case of G-graded modules. The main results of the paper is Theorem 3.9 and Corollary 3.11, where, for any subgroup H of G, we prove the existence of equivalences between certain subcategories of G/H-graded R-modules associated to the G-graded Σ-quasiprojective module M, and a subcategory of G/H-graded E-modules; we investigate the compatibility of these equivalences with the above mentioned grade forgetting functor.

Note that if $E \neq S$, Theorem 3.9 does not immediately give the equivalences of García Hernández and Gómez Pardo. The precise relationship with their results is presented in Theorem 3.12. In the last section we discuss applications of our results to various particular cases.

Let us briefly present our general assumptions and notations. Rings are associative with identity, and modules are left, unless otherwise stated. A module will often be regarded as a right module over its endomorphism ring, so if f and g are two composable homomorphisms, we shall write $fg = g \circ f$.

If A is a ring, we denote by A-Mod the category of A-modules, and by $\mathcal{L}(A)$ the lattice of left ideals of A. If $I \in \mathcal{L}(A)$ and $a \in A$, let $(I : a) = \{b \in A \mid ba \in I\}$, and if M is a A-module and $m \in M$, then $\ell_A(m) = \{a \in A \mid am = 0\}$ is the left annihilator of m.

If $R = \bigoplus_{g \in G} R_g$ is a ring graded by a group G, and H is a subgroup of G, we denote by $(G/H, R)$-Gr the category of R-modules graded by the G-set G/H and grade-preserving R-homomorphisms. This is a Grothendieck category, and we refer the reader to [14] for its basic properties. In particular, $(G/1, R)$-Gr is the category R-Gr of G-graded R-modules, and $(G/G, R)$-Gr $= R$-Mod. If $N = \bigoplus_{x \in G/H} N_x$ is an object of $(G/H, R)$-Gr, and X is a subset of G/H, we denote $N_X = \bigoplus_{x \in X} N_x$. We shall also use the notation $[G/H]$ (respectively $[H\backslash G]$) for a system of representatives of the left (respectively right) cosets of H in G.

We refer to [19] for general facts on torsion theory, and to [17] for the theory of group graded rings. All the other notation used in the paper will be introduced or recalled in Sections 1 and 2.

1. Preliminaries

1.1. Torsion theory. Let $\mathcal{A}$ be a Grothendieck category. A class $\mathcal{T}$ of objects of $\mathcal{A}$ is called a *pretorsion class* if it is closed under quotients and direct sums. We denote by $\mathbf{t}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ the *preradical* associated to $\mathcal{T}$, so for every object M of $\mathcal{A}$, $\mathbf{t}_{\mathcal{A}}(M)$ is the largest subobject of M belonging to $\mathcal{T}$. A class $\mathcal{T}$ (respectively $\mathcal{F}$) is called *torsion* (respectively *torsionfree*) *class* if it is closed under quotients, extensions and direct sums (respectively subobjects, extensions and products). A class which is both torsion and torsionfree is said to be a *TTF-class*. The torsion theory $(\mathcal{T}, \mathcal{F})$ is *hereditary* if $\mathcal{T}$ is closed under subobjects too. Recall that $M \in \mathcal{T}$ if and only if $\mathbf{t}_{\mathcal{A}}(M) = M$ and $M \in \mathcal{F}$ if and only if $\mathbf{t}_{\mathcal{A}}(M) = 0$.

1.2. Localizing subcategories and Gabriel's theorem. The full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *closed subcategory* (or hereditary pretorsion class) if it closed under subobjects, quotients and direct sums; if this is the case $\mathcal{C}$ is a Grothendieck category too. $\mathcal{C}$ is a *localizing subcategory* if in addition it it closed under extensions. If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ then one may construct the quotient category $\mathcal{A}/\mathcal{C}$ with the canonical functors:

$$\mathcal{A} \underset{\mathbf{i}_{\mathcal{C}}}{\overset{\mathbf{a}_{\mathcal{C}}}{\rightleftarrows}} \mathcal{A}/\mathcal{C}$$

which satisfy the following properties:
- $\mathbf{a}_{\mathcal{C}}$ is exact and $\mathcal{C} = \operatorname{Ker} \mathbf{a}_{\mathcal{C}}$;
- $\mathbf{i}_{\mathcal{C}}$ is a full and faithful right adjoint of $\mathbf{a}_{\mathcal{C}}$;
- The natural transformation $\Phi \colon \mathbf{a}_{\mathcal{C}} \circ \mathbf{i}_{\mathcal{C}} \to \mathbf{1}_{\mathcal{A}/\mathcal{C}}$ is an isomorphism.

Consider the natural transformation $\Psi \colon \mathbf{1}_{\mathcal{A}} \to \mathbf{i}_{\mathcal{C}} \circ \mathbf{a}_{\mathcal{C}}$. Then for any object M of $\mathcal{A}$ the kernel and the cokernel of Ψ_M are $\mathcal{C}$-torsion (belong to $\mathcal{C}$), and M is called $\mathcal{C}$-*closed* if Ψ_M is an isomorphism. Recall that M is $\mathcal{C}$-closed if and only if it is $\mathcal{C}$-torsionfree and $\mathcal{C}$-*injective* (that is, for each short exact sequence $0 \to N' \xrightarrow{u} N \to \operatorname{Coker} u \to 0$ with $\operatorname{Coker} u \in \mathcal{C}$ the induced homomorphism $u^* \colon \operatorname{Hom}_{\mathcal{A}}(N, M) \to \operatorname{Hom}_{\mathcal{A}}(N', M)$ is surjective).

Moreover, the quotient category $\mathcal{A}/\mathcal{C}$ can be identified with the full subcategory of $\mathcal{A}$ consisting of all $\mathcal{C}$-closed objects, which is again a Grothendieck category.

Conversely, a theorem of Gabriel states that if $\mathbf{a} \colon \mathcal{A} \to \mathcal{A}'$ and $\mathbf{i} \colon \mathcal{A}' \to \mathcal{A}$ are functors between Grothendieck categories such that $\mathbf{a}$ is an exact left adjoint of $\mathbf{i}$ and the associated natural transformation $\Phi \colon \mathbf{a} \circ \mathbf{i} \to \mathbf{1}_{\mathcal{A}'}$ is an isomorphism, then $\operatorname{Ker} \mathbf{a}$ is a localizing subcategory of $\mathcal{A}$ and $\mathbf{a}$ induces an equivalence between $\mathcal{A}/\operatorname{Ker} \mathbf{a}$ and $\mathcal{A}'$.

1.3. Linear and Gabriel topologies. If $\mathcal{A} = A$-Mod for a ring A, then the closed (respectively localizing) subcategories of $\mathcal{A}$ correspond to the left linear (respectively Gabriel) topologies $\mathcal{G}$ of A. Recall that a filter $\mathcal{G}$ of left ideals of A is a *left linear topology* if $I \in \mathcal{G}$ and $a \in A$ implies $(I : a) \in \mathcal{G}$ and is a *left Gabriel topology* if, in addition, for each $I \in \mathcal{L}(A)$ for that there is an $I' \in \mathcal{G}$ with $(I : a) \in \mathcal{G}$ for all $a \in I'$ we have $I \in \mathcal{G}$. The correspondence is given by $\mathcal{C} \mapsto \mathcal{G}_{\mathcal{C}} = \{I \in_A A \mid A/I \in \mathcal{C}\}$ and $\mathcal{G} \mapsto \mathcal{C}_{\mathcal{G}} = \{X \in A\text{-Mod} \mid \ell_A(x) \in \mathcal{G} \text{ for all } x \in X\}$. In this case we shall write $\mathcal{G}$-torsion

(free), $\mathcal{G}$-injective and $\mathcal{G}$-closed instead of $\mathcal{C}$-torsion (free), $\mathcal{C}$-injective and $\mathcal{C}$-closed, and $(A, \mathcal{G})$-Mod denotes the full subcategory of A-Mod consisting of $\mathcal{G}$-closed modules.

1.4. Pure ideals. The following interesting situation was discussed in [2]. Let J be an two-sided ideal and assume that J_A is *pure* (that is, for each $M \in A$-Mod the canonical morphism $J \otimes_A M \to JM$ is an monomorphism) or, equivalently, A/J is flat as right A-module. It follows that J is an idempotent ideal. Consider the full subcategories of A-Mod:

$$\mathcal{J} = \{M \in A\text{-Mod} \mid JM = M\}, \qquad \mathcal{C} = \{M \in A\text{-Mod} \mid JM = 0\}$$

and the functors

$$\mathcal{J} \underset{\mathbf{a}}{\overset{\mathbf{i}}{\rightleftarrows}} A\text{-Mod} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} A/J\text{-Mod}$$

where $\varphi^*(M) = A/J \otimes_A M \cong M/JM$, φ_* is the scalar restriction, $\mathbf{i}(M) = \operatorname{Hom}_A(J, M)$ and $\mathbf{a}(M) = J \otimes_A M \cong JM$. Thus we have:

a) $\mathcal{J}$ is a localizing subcategory and $\mathcal{C}$ is a TTF-class;

b) φ^* and φ_* induce equivalences A-Mod$/\mathcal{J} \cong A/J$-Mod $\cong \mathcal{C} = \operatorname{Ker} \mathbf{a}$;

c) $\mathbf{a}$ is the right adjoint of the inclusion functor $\mathbf{j} \colon \mathcal{J} \to A$-Mod and $\mathbf{a}$ a left adjoint of $\mathbf{i}$;

d) the natural transformation $\Phi \colon \mathbf{a} \circ \mathbf{i} \to \mathbf{1}_{\mathcal{J}}$ is an isomorphism, hence $\mathbf{a}$ and $\mathbf{i}$ induce an equivalence A-Mod$/\mathcal{C} \cong \mathcal{J}$.

As a particular case, let $J = \sum_{\lambda \in \Lambda} p_\lambda A$ where $\{p_\lambda \mid \lambda \in \Lambda\}$ is a non-empty set of orthogonal idempotents of A. Then J is a pure ideal and assume that J is a two-sided ideal, that is, $\sum_{\lambda \in \Lambda} A p_\lambda \subseteq J$. (Note that if in addition $\sum_{\lambda \in \Lambda} A p_\lambda = J$, then J is a ring with enough idempotents.) Then $\mathcal{J}$ is isomorphic to category J-Mod of unital J-modules (that is $JN = N$).

1.5. Subcategories associated to an object. Return to our general situation, and fix an object M of $\mathcal{A}$. Let $\operatorname{Gen}[M]$ be the full subcategory of $\mathcal{A}$ consists of M-generated objects and $\boldsymbol{\sigma}[M]$ the full subcategory consisting of M-subgenerated objects (subobjects of objects of $\operatorname{Gen}[M]$), so $\boldsymbol{\sigma}[M]$ is the smallest closed subcategory of $\mathcal{A}$ containing M. If $N \in \mathcal{A}$ let $N_M = \operatorname{Tr}_M(N)$ be the largest M-generated subobject of N. If M generates $\boldsymbol{\sigma}[M]$, that is, $\boldsymbol{\sigma}[M] = \operatorname{Gen}[M]$, then M is called *self-generator*.

Let $\mathcal{T}$ be the smallest localizing subcategory of $\boldsymbol{\sigma}[M]$ containing the objects N/N_M with $N \in \boldsymbol{\sigma}[M]$. Denote $\mathbf{t} \colon \boldsymbol{\sigma}[M] \to \boldsymbol{\sigma}[M]$ the corresponding radical, $\overline{N} = N/\mathbf{t}(N)$ and $\mathcal{F}$ the corresponding torsion-free class. The objects of $\mathcal{F}$ are called M-*faithful* or M-*distinguished*, and it is easy to see that $N \in \boldsymbol{\sigma}[M]$ is M-faithful if and only if for every non-zero morphism $g \colon X \to N$ in $\boldsymbol{\sigma}[M]$ there is a morphism $f \colon M \to X$ such that $g \circ f \neq 0$ [8, Proposition 1.2].

We shall consider the following full subcategories of $\boldsymbol{\sigma}[M]$:

• $\operatorname{Pres}[M]$ consisting of all M-presented objects of $\mathcal{A}$;

• $\mathcal{C}[M] = \boldsymbol{\sigma}[M]/\mathcal{T}$ which can be identified as usual with the full subcategory of $\mathcal{T}$-closed objects;

- GF[M] consisting of all M-generated, M-faithful objects.

1.6. Σ-quasiprojective objects. An object M of $\mathcal{A}$ is N-*projective* if for any exact sequence $N \rightarrow N' \rightarrow 0$ the sequence $\mathrm{Hom}_{\mathcal{A}}(M, N) \rightarrow \mathrm{Hom}_{\mathcal{A}}(M, N') \rightarrow 0$ is exact too. M is called *quasiprojective* if it is M-projective and M is Σ-*quasiprojective* if it is $M^{(\Lambda)}$-projective for any set Λ.

Assume that M is Σ-quasiprojective. Then the hereditary torsion class $\mathcal{T}$ of 1.5 consists of objects $N \in \boldsymbol{\sigma}[M]$ satisfying $\mathrm{Hom}_{\mathcal{A}}(M, N) = 0$, and it is actually a TTF-class. The corresponding torsion class is $\{X \in \boldsymbol{\sigma}[M] \mid \mathrm{Hom}_{\mathcal{A}}(X, N) = 0 \text{ for all } N \in \mathcal{T}\}$, which coincides with $\mathrm{Gen}[M]$ since $X/X_M \in \mathcal{T}$ for all $X \in \boldsymbol{\sigma}[M]$. It follows that $X \mapsto X_M$ is the radical associated to the torsion class $\mathrm{Gen}[M]$. For the corresponding torsionfree class $\mathcal{F}$ we have by [7, Proposition 1.2] that $N \in \boldsymbol{\sigma}[M]$ is M-faithful if and only if $\mathrm{Hom}_{\mathcal{A}}(M, X) \neq 0$ for every nonzero subobject X of N, or equivalently, $X_M \neq 0$ for every nonzero subobject X of N.

Finally, by [6, Proposition 1.2] we have that $\mathbf{a}(M)$ is a projective generator of $\mathcal{C}[M]$ and $\mathrm{End}_{\mathcal{A}}(M) \cong \mathrm{End}_{\mathcal{C}[M]}(\mathbf{a}(M))$, where $\mathbf{a} \colon \mathcal{A} \rightarrow \mathcal{C}[M]$ is the canonical functor.

1.7. Functors between categories of modules graded by G-sets. Let G be a group, $R = \bigoplus_{g \in G} R_g$ a G-graded ring and fix two subgroups $K \leq H$ of G. We have two functors connecting the categories $(G/H, R)$-Gr and $(G/K, R)$-Gr.

The *grade forgetting* functor

$$\mathbf{U} = \mathbf{U}_{G/H}^{G/K} \colon (G/K, R)\text{-Gr} \rightarrow (G/H, R)\text{-Gr}$$

is defined as follows: for $M = \bigoplus_{x \in G/K} M_x \in (G/K, R)$-Gr let

$$\mathbf{U}(M) = \bar{M} = \bigoplus_{y \in G/H} \bar{M}_y,$$

where $\bar{M} = M$ and $\bar{M}_y = \bigoplus_{x \subseteq y} M_x$ for all $y \in G/H$, and $\mathbf{U}(f) = f$ for every morphism $f \colon M \rightarrow M'$ in $(G/K, R)$-Gr. In most of the cases, when it will be clear from context, we simply denote $\mathbf{U}(M)$ by M.

The functor $\mathbf{U}$ has a right adjoint

$$\mathbf{F} = \mathbf{F}_{G/H}^{G/K} \colon (G/H, R)\text{-Gr} \rightarrow (G/K, R)\text{-Gr}$$

defined as follows: for $N = \bigoplus_{y \in G/H} N_y \in (G/H, R)$-Gr let

$$\mathbf{F}(N) = \tilde{N} = \bigoplus_{x \in G/K} \tilde{N}_{xy},$$

where $\tilde{N}_x = N_{xH}$ with multiplication by scalars given by $r_g \tilde{n}_x = r_g n_y \in N_{gx}$, for $y = xH$, $\tilde{n}_x = n_y \in N_y$, $r_g \in R_g$, $g \in G$. If $f \colon N \rightarrow N'$ is morphism in $(G/H, R)$-Gr, then $\tilde{f} = \mathbf{F}(f) \colon \tilde{N} \rightarrow \tilde{N}'$ is given by $\tilde{f}(\tilde{n}_x) = f(n_y) \in \tilde{N}_x = N_y$, with $y = xH$ and $\tilde{n}_x = n_y$ as above.

The unit ζ if the adjoint pair $(\mathbf{U}, \mathbf{F})$ is defined by

$$\zeta_M \colon M \to \mathbf{F}(\mathbf{U}(M)), \ \zeta_M(m_x) = m_x \in \mathbf{F}(\mathbf{U}(M))_x$$

for all $x \in G/H$ and $m_x \in M_x$. The counit ξ is given by

$$\xi_N \colon \mathbf{U}(\mathbf{F}(N)) \to N, \ \xi(\tilde{n}_x) = \tilde{n}_x \in N_{xH}$$

for all $\tilde{n}_x = n_y \in N_y$, $x \in G/K$ and $y = xH$. Observe that ξ_N is an epimorphism and ζ_M is a monomorphism for every $N \in (G/H, R)$-Gr and $M \in (G/K, R)$-Gr.

Recall also that if H/K is finite then $\mathbf{F}_{G/H}^{G/K}$ is a left adjoint of $\mathbf{U}_{G/H}^{G/K}$ too.

If $x \in G/H$ one can also define the x-th *suspension* functor

$$\mathbf{S}^x \colon (G/H, R)\text{-Gr} \to (G/^xH, R)\text{-Gr}, \ \mathbf{S}^x(N) = N(x)$$

where $^xH = xHx^{-1}$, $N(x) = N$ and $N(x)_y = N_{yx}$ for all $y \in G/^xH$. Clearly $\mathbf{S}^x$ is an equivalence with inverse $\mathbf{S}^{x^{-1}}$.

The following facts are well-know in the case of R-Gr and R-Mod see [16, Proposition 1.4]:

1.8. Lemma. a) *The functors* $\mathbf{U}_{G/H}^{G/K}$ *and* $\mathbf{F}_{G/H}^{G/K}$ *are exact and commute with direct products and direct sums;*

b) *If* $K = 1$ *and* $M \in R$-Gr *then* $\mathbf{F}_{G/H}^{G/K}(\mathbf{F}_{G/H}^{G/K}(M)) \cong \bigoplus_{h \in H} M(h)$ *as* G-*graded* R-*module.*

Proof. a) It is clear that $\mathbf{U}$ and $\mathbf{F}$ are exact, commute with direct product and $\mathbf{U}$ commute with direct sums.

Let $N = \bigoplus_{\lambda \in \Lambda} N^\lambda$ be a direct sum in $(G/H, R)$-Gr and let $q_\lambda \colon N^\lambda \to N$ be the canonical monomorphism. We have the morphism $\mathbf{F}(q_\lambda) \colon \mathbf{F}(N^\lambda) \to \mathbf{F}(N)$ in $(G/K, R)$-Gr which induce the morphism

$$u \colon \bigoplus_{\lambda \in \Lambda} \mathbf{F}(N^\lambda) \to \mathbf{F}(N)$$

Let $(\tilde{n}^\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} \mathbf{F}(N^\lambda)$ be a homogeneous element of degree $x \in G/K$ where for each $\lambda \in \Lambda$, $\tilde{n}^\lambda = n^\lambda \in N_y^\lambda$ for $y \in G/H$ such that $x \subseteq y$. Then $u((\tilde{n}^\lambda)_{\lambda \in \Lambda}) = \tilde{n} \in \mathbf{F}(N)_x$, with $\tilde{n} = n \in N_y$ where $n = (\tilde{n}^\lambda)_{\lambda \in \Lambda}$. It is easy to see that u is an isomorphism.

b) Denote $\mathbf{U} = \mathbf{U}_{G/H}^{G/K}$, $\mathbf{F} = \mathbf{F}_{G/H}^{G/K}$ and $N = \mathbf{U}(M) = \bigoplus_{g \in [G/H]} N_{gH}$, where $N_{gH} = \bigoplus_{h \in H} M_{gh}$.

Let $h \in H$ and $m \in M(h)_g = M_{gh}$. By the above, m determines an unique element $\phi(m) = m$ belonging to the component M_{gh} of $\tilde{N}_g$. It is clear that we have obtained an isomorphism $\phi \colon \bigoplus_{h \in H} M(h) \to \tilde{N}$ of G-graded R-module.

1.9. Graded groups of homomorphisms. Let $M = \bigoplus_{g \in G} M_g \in R$-Gr, and $N = \bigoplus_{x \in G/H} N_x \in (G/H, R)$-Gr. By [11, 2.9], for each $x \in G/H$ the set

$$\mathrm{HOM}_{G/H,R}(M, N)_x = \{f \in \mathrm{Hom}_R(M, N) \mid f(M_g) \subseteq N_{gx} \ \text{for all} \ g \in G\}$$

is an additive subgroup of $\mathrm{Hom}_R(M,N)$, and by [12, Proposition 3.4] it is a closed subset of $\mathrm{Hom}_R(M,N)$ in the finite topology. Note that $\mathrm{HOM}_{G/H,R}(M,N)_H = \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(M,N)$. Moreover the sum

$$\mathrm{HOM}_{G/H,R}(M,N) = \sum_{x \in G/H} \mathrm{HOM}_{G/K,R}(M,N)_x$$

is direct, and by [12, Theorem 3.7], $\mathrm{Hom}_R(M,N)$ is the completion of $\mathrm{HOM}_{G/H,R}(M,N)$ in the finite topology.

In particular, if $M' \in R\text{-}\mathrm{Gr}$, we denote $\mathrm{HOM}_R(M,N) = \mathrm{HOM}_{G/1,R}(M,N)$, and let $E = \mathrm{END}(M)^{\mathrm{op}} = \mathrm{HOM}_R(M,M)$ which is a subring of $S = \mathrm{End}_R(M)^{\mathrm{op}}$. Then E is a G-graded ring, M becomes a G-graded (R,E)-bimodule and $\mathrm{HOM}_{G/H,R}(M,N)$ a G/H-graded E-module.

Remark that for $x = gH$ we have the equalities

$$\mathrm{HOM}_{G/H,R}(M,N)_x = \mathrm{Hom}_{(G/^gH,R)\text{-}\mathrm{Gr}}(\mathbf{U}^{G/1}_{G/H}(M),N(gH))$$
$$= \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\mathbf{U}^{G/1}_{G/H}(M(g^{-1})),N).$$

Indeed, the first equality is in [11, 2.9]. For the second, if $f \in \mathrm{Hom}_R(M,N)$ then we have the logical equivalences:

$$f(M_h) \subseteq N_{ghH} \text{ for all } h \in G \Leftrightarrow f(M_{g'g^{-1}}) \subseteq N_{g'H} \text{ for all } g' \in G, \ (g' = gh)$$
$$\Leftrightarrow f(\bigoplus_{k \in g'H} M_{kg^{-1}}) \subseteq N_{g'H} \text{ for all } g' \in G$$
$$\Leftrightarrow f(\bigoplus_{k \in g'H} M(g^{-1})_k) \subseteq N_{g'H} \text{ for all } g' \in G$$
$$\Leftrightarrow f(\mathbf{U}^{G/1}_{G/H}(M(g^{-1}))_{g'H}) \subseteq N_{g'H} \text{ for all } g' \in G$$
$$\Leftrightarrow f \in \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\mathbf{U}^{G/1}_{G/H}(M(g^{-1})),N).$$

1.10. Static modules. The functor

$$\mathrm{HOM}_{G/H,R}(M,-)\colon (G/H,R)\text{-}\mathrm{Gr} \to (G/H,E)\text{-}\mathrm{Gr}$$

is a right adjoint of the functor $M \otimes_E -$. The unit and the counit of adjunction are defined by

$$\eta^{G/H}_X\colon X \to \mathrm{HOM}_{G/H,R}(M,M \otimes_E X), \quad \eta^{G/H}_X(x)(m) = x \otimes m,$$
$$\rho^{G/H}_N\colon M \otimes_E \mathrm{HOM}_{G/H,R}(M,N) \to N, \quad m \otimes f \mapsto mf = f(m)$$

for all $X \in (G/H,E)\text{-}\mathrm{Gr}$ and $N \in (G/H,R)\text{-}\mathrm{Gr}$.

We shall consider the full subcategories:

$$\mathrm{Stat}^{G/H}[M] = \{N \in (G/H,R)\text{-}\mathrm{Gr} \mid \rho^{G/H}_N \text{ is an isomorphism}\}$$

$$\mathrm{Adst}^{G/H}[M] = \{X \in (G/H, E)\text{-Gr} \mid \eta_X^{G/H} \text{ is an isomorphism}\}$$

of $(G/H, R)$-Gr and $(G/H, E)$-Gr respectively.

Similarly the functor

$$\mathrm{Hom}_{(G/H,R)\text{-Gr}}(M, -) \colon (G/H, R)\text{-Gr} \to E_H\text{-Mod}$$

is a right adjoint of $M \otimes_{E_H} - \colon E_H\text{-Mod} \to (G/H, R)\text{-Gr}$, where $E_H = \bigoplus_{h \in H} E_h$ is a subring of $\mathrm{End}_{(G/H,R)\text{-Gr}}(M)^{\mathrm{op}}$.

1.11. Lemma. *Let M be a G-graded (R, E)-module and I a G-graded E-module. Then the functors*

$$\mathrm{HOM}_{G/H,R}(M \otimes_E I, -), \quad \mathrm{HOM}_{G/H,E}(I, \mathrm{HOM}_{G/H,R}(M, -)) \colon (G/H, R)\text{-Gr} \to \mathcal{A}b$$

are naturally isomorphic.

Proof. Since $\mathrm{HOM}_{G/H,R}(M, -)$ is the right adjoint of $M \otimes_E -$, for any $g \in G$ and $N \in (G/H, R)$-Gr we have the natural isomorphism

$$\mathrm{Hom}_{(G/H,R)\text{-Gr}}(M \otimes_E I(g^{-1}), N) \cong \mathrm{Hom}_{(G/H,E)\text{-Gr}}(I(g^{-1}), \mathrm{HOM}_{G/H,R}(M, N)).$$

Let $[G/H]$ a set of representatives for the left cosets of H in G. We obtain the natural isomorphisms:

$$\begin{aligned}
\mathrm{HOM}_{G/H,R}(M \otimes_E I, N) &\cong \bigoplus_{g \in [G/H]} \mathrm{Hom}_{(G/H,R)\text{-Gr}}(M \otimes_E I(g^{-1}), N) \\
&\cong \bigoplus_{g \in [G/H]} \mathrm{Hom}_{(G/H,E)\text{-Gr}}(I(g^{-1}), \mathrm{HOM}_{G/H,R}(M, N)) \\
&\cong \mathrm{HOM}_{G/H,E}(I, \mathrm{HOM}_{G/H,R}(M, N)).
\end{aligned}$$

1.12. The grade forgetting functor and HOM. Let $K \leq H \leq G$, $M \in R$-Gr and $N \in (G/H, R)$-Gr. By [12, Corollary 3.8 a)], $\mathrm{HOM}_{G/K,R}(M, N)$ is a dense subset of $\mathrm{HOM}_{G/K,R}(M, N)$ in the finite topology, and it is an interesting question whether

$$\mathrm{HOM}_{G/H,R}(M, \mathbf{U}_{G/H}^{G/K}(N)) = \mathbf{U}_{G/H}^{G/K}(\mathrm{HOM}_{G/H,R}(M, N)).$$

The equality clearly holds if the set H/K is finite. If H/K is infinite then by [12, Theorem 4.9] holds for every $N \in (G/H, R)$-Gr if and only if M is small in R-Mod.

1.13. Strongly graded rings. The G-graded ring R is called *strongly graded* if $R_g R_h = R_{gh}$ for all $g, h \in G$. By a theorem of E. Dade, R is strongly graded if and only if the functors $R \otimes_{R_1} - \colon R_1$-Mod $\to R$-Gr and $(-)_1 \colon R$-Gr $\to R_1$-Mod are inverse equivalences of categories. In this case, the functors $R \otimes_{R_H} - \colon R_H$-Mod $\to (G/H, R)$-Gr and $(-)_H \colon (G/H, R)$-Gr $\to R_H$-Mod are also inverse equivalences.

If R is strongly graded, then a two-sided ideal I_1 of R_1 is the 1-component of a G-graded two-sided ideal I of R if and only if I_1 is G-*invariant*, that is $R_g I_1 R_{g^{-1}} = I_1$ for all $g \in G$.

Recall also that if $M \in R$-Gr then $E = \mathrm{END}_R(M)^{op}$ is strongly graded if and only if M is *weakly G-invariant*, that is, $M(g)$ is a direct summand in R-Gr of a finite direct sum of copies of M for all $g \in G$.

2. Rigid subcategories of $(G/H, R)$-Gr and graded Gabriel topologies

2.1. Let H be a subgroup of G.

A class $\mathcal{D} \subseteq (G/H, R)$-Gr will be called *rigid* if, for every $N \in \mathcal{D}$, $\mathbf{U}(\mathbf{F}(N)(g)) \in \mathcal{D}$ for all $g \in G$. (Clearly, here $\mathbf{U} = \mathbf{U}_{G/H}^{G/1}$ and $\mathbf{F} = \mathbf{F}_{G/H}^{G/1}$). In the case of R-Gr, this concept was introduced in [10, Section 2].

If $\mathcal{C} \in R$-Gr is a pretorsion class, we denote by $\mathcal{C}^{G/H}$ the smallest pretorsion class of $(G/H, R)$-Gr which contains the objects $\mathbf{U}(M)$ for $M \in \mathcal{C}$.

If $\mathcal{D} \subseteq (G/H, R)$-Gr is a pretorsion free class (that is, it is closed under subobjects and direct products) we denote by $\mathcal{D}^{\mathrm{gr}}$ the smallest pretorsion class R-Gr containing the objects $\mathbf{F}(N)$ for $N \in \mathcal{D}$.

2.2. Proposition. *Let $\mathcal{C} \subseteq R$-Gr be a rigid pretorsion class*

a) *We have the equalities:*

$$\mathcal{C}^{G/H} = \{N \in (G/H, R)\text{-Gr} \mid \text{ there is an epimorphism } \mathbf{U}(M) \to N \text{ for some } M \in \mathcal{C}\}$$
$$= \{N \in (G/H, R)\text{-Gr} \mid \mathbf{F}(N) \in \mathcal{C}\};$$

b) *If $\mathcal{C}$ is a closed (respectively localizing) subcategory then $\mathcal{C}^{G/H}$ is also a closed (respectively localizing) subcategory.*

Proof. a) Denote by $\mathcal{D}$ and $\mathcal{D}'$ the classes defined above. If $N \in \mathcal{D}$ then there is $M \in \mathcal{C}$ and an epimorphism $\mathbf{U}(M) \to N$. Then $N \in \mathcal{C}^{G/H}$, since $\mathbf{U}(M) \in \mathcal{C}^{G/H}$ and $\mathcal{C}^{G/H}$ is closed under epimorphic images.

Conversely, using the fact that $\mathbf{U}$ is exact and commute with direct sums, it is easy to see that $\mathcal{D}$ is a pretorsion class, containing $\mathbf{U}(M)$, $M \in \mathcal{C}$, hence $\mathcal{C}^{G/H} \subseteq \mathcal{D}$.

If $N \in \mathcal{D}'$ then $\mathbf{F}(N) \in \mathcal{C}$ and we have the epimorphism $\xi_N \colon \mathbf{U}(\mathbf{F}(N)) \to N$, hence $N \in \mathcal{D}$.

Conversely, if $N \in \mathcal{D}$ then there is $M \in \mathcal{C}$ and an epimorphism $\mathbf{U}(M) \to N$, and also an epimorphism $\mathbf{F}(\mathbf{U}(M)) \to \mathbf{F}(N)$ in R-Gr. But $\mathbf{F}(\mathbf{U}(M)) = \bigoplus_{h \in H} M(h)$ belongs to $\mathcal{C}$ since $\mathcal{C}$ is rigid and closed under direct sums, hence $\mathbf{F}(N) \in \mathcal{C}$. Finally, $\mathcal{C}^{G/H}$ is rigid, since if $N \in \mathcal{C}^{G/H}$ then $\mathbf{F}(N)(g) \in \mathcal{C}$ for all $g \in G$, hence $\mathbf{U}(\mathbf{F}(N)(g)) \in \mathcal{C}^{G/H}$.

b) follows immediately from a) and the exactness of $\mathbf{F}$.

2.3. Proposition. *Let $\mathcal{D} \subseteq (G/H, R)$-Gr be a rigid pretorsionfree class.*

a) *We have the equalities:*

$$\mathcal{D}^{\mathrm{gr}} = \{M \in R\text{-Gr} \mid \text{ there is an monomorphism } M \to \mathbf{F}(M) \text{ for some } M \in \mathcal{D}\}$$
$$= \{M \in R\text{-Gr} \mid \mathbf{U}(M) \in \mathcal{D}\};$$

b) *If $\mathcal{D}$ is a closed (respectively localizing) subcategory then $\mathcal{D}^{G/H}$ is also a closed (respectively localizing) subcategory.*

Proof. a) Denote by $\mathcal{C}$ and $\mathcal{C}'$ the classes defined above. If $M \in \mathcal{D}$ then there is $N \in \mathcal{D}$ and an monomorphism $M \to \mathbf{F}(N)$. Then $M \in \mathcal{D}^{\mathrm{gr}}$, since $\mathbf{F}(N) \in \mathcal{D}^{\mathrm{gr}}$ and $\mathcal{D}^{\mathrm{gr}}$ is closed under subobjects.

Conversely, since $\mathbf{F}$ is exact and commute with direct products, it follows easily that $\mathcal{C}$ is a pretorsionfree class, containing $\mathbf{F}(N)$, $M \in \mathcal{D}$, hence $\mathcal{D}^{\mathrm{gr}} \subseteq \mathcal{C}$.

If $M \in \mathcal{C}'$ then $\mathbf{U}(M) \in \mathcal{D}$ and $\mathbf{F}(\mathbf{U}(M)) \in \mathcal{D}^{\mathrm{gr}}$. Since the $\zeta_M \colon M \to \mathbf{F}(\mathbf{U}(M))$ is a monomorphism, we deduce $M \in \mathcal{C}$.

Conversely, if $M \in \mathcal{C}$ then there is monomorphism $M \to \mathbf{F}(N)$ for some $N \in \mathcal{D}$ and also a monomorphism $\mathbf{U}(M) \to \mathbf{U}(\mathbf{F}(N))$ in $(G/H, R)$-Gr. But $\mathbf{U}(\mathbf{U}(N)) \in \mathcal{D}$ since $\mathcal{C}$ is rigid, hence $\mathbf{U}(M) \in \mathcal{D}$.

To prove that $\mathcal{D}^{\mathrm{gr}}$ is rigid, let $M \in \mathcal{D}^{\mathrm{gr}}$ and $g \in G$. There is a monomorphism $M \to \mathbf{F}(N)$ for some $N \in \mathcal{D}$, hence a monomorphism $M(g) \to \mathbf{F}(N)(g)$. Let $M' = \mathbf{F}(N)(g)$ and $N' = \mathbf{U}(M')$. Then $N' \in \mathcal{D}$ since $\mathcal{D}$ is rigid and we have the monomorphism $\zeta_{M'} \colon M' \to \mathbf{F}(\mathbf{U}(M')) = \mathbf{F}(N')$. It follows that we have a monomorphism $M(g) \to \mathbf{F}(N')$ in R-Gr, hence $M(g) \in \mathcal{D}^{\mathrm{gr}}$.

b) follows immediately from a) and the exactness of $\mathbf{U}$.

2.4. Corollary. a) *If $\mathcal{C}$ is a rigid closed subcategory of R-Gr then $(\mathcal{C}^{G/H})^{\mathrm{gr}} = \mathcal{C}$.*

b) *If $\mathcal{D}$ is a rigid closed subcategory of $(G/H, R)$-Gr then $(\mathcal{D}^{\mathrm{gr}})^{G/H} = \mathcal{D}$*

2.5. G/H-graded ideals. In order to associate G/H-graded linear topologies to rigid closed subcategories of $(G/H, R)$-Gr, we have to see what a G/H-graded ideal of R should be.

Let N be a G/H-graded R-module and $n \in N_{gH}$. Then the map $\mathbf{U}_{G/H}^{G/1}(R(g^{-1}) \to N \ r \mapsto rn$ is a homomorphism of G/H-graded R-module, so its kernel is a G/H-graded submodule of $R(g^{-1})$. Observe also that if $g, g' \in G$ then $\mathbf{U}_{G/H}^{G/1}(R(g^{-1})) = \mathbf{U}_{G/H}^{G/1}(R(g'^{-1}))$ in $(G/H, R)$-Gr if and only if $Hg = Hg'$.

Denote
$$\mathcal{L}^{G/H}(R) = \{\mathcal{L}_{Hg}^{G/H}(R) \mid g \in [H\backslash G]\}$$
where $\mathcal{L}_{Hg}^{G/H}(R)$ is the lattice of G/H-graded submodules of $R(g)$.

We shall write $\mathcal{H} \subseteq \mathcal{L}^{G/H}(R)$ if $\mathcal{H} = \{\mathcal{H}_{Hg} \mid g \in [H\backslash G]\}$ and $\mathcal{H}_{Hg} \subseteq \mathcal{L}_{Hg}^{G/H}(R)$ for all $g \in [H\backslash G]$, similarly $I \in \mathcal{H}$ means that there is an $g \in G$ such that $I \in \mathcal{H}_{Hg}$.

Let $I \in \mathcal{L}_{Hg}^{G/H}(R)$ and $r \in R(g')_{\sigma H} = R_{\sigma Hg'}$. Then the definition $(I : r) = \{b \in R \mid br \in I\}$ makes sense if and only if $Hg = Hg'$ and in this case it is clear that $(I : r) \in \mathcal{L}_{H\sigma^{-1}}^{G/H}(R)$, since $(I : r) = \ell_R(r + I)$ in $R(g)/I$.

2.6. Rigidity. Next we show that if we forget the grading, then the sets $\mathcal{L}_{Hg}^{G/H}(R)$ are actually equal. More precisely, for each $g \in G$ we define a bijection from $\mathcal{L}_H^{G/H}(R)$ to $\mathcal{L}_{Hg}^{G/H}(R)$, sending I to I^g, where $I = I^g$ as subsets of R.

Let $I \in \mathcal{L}_H^{G/H}(R)$ and $g \in G$. Then $R/I \in (G/H, R)$-Gr and $I = \ell_R(1 + I)$. We consider the G/H-graded R-module
$$N = \mathbf{U}_{G/H}^{G/1}(\mathbf{F}_{G/H}^{G/1}(R/I)(g)) = \bigoplus_{\sigma \in [G/H]} N_{\sigma H}$$

where $N_{\sigma H} = \bigoplus_{h \in H}(R/I)_{\sigma hgH}$. Setting $\sigma = g^{-1}$, we see that $1 + I$ appears as an element of degree $g^{-1}H$ of N, so its left annihilator $I^g = I$ is an element of $\mathcal{L}_{Hg}^{G/H}(R)$.

If $\mathcal{H} \subseteq \mathcal{L}^{G/H}(R)$ we say that $\mathcal{H}$ is *rigid* if for each $g \in G$ the correspondence $I \mapsto I^g$ gives a bijection from $\mathcal{H}_H$ to $\mathcal{H}_{Hg}$.

2.7. Graded Gabriel topologies. Let $\mathcal{H} = \{\mathcal{H}_{Hg} \mid g \in [H\backslash G]\} \subseteq \mathcal{L}^{G/H}(R)$ with $\mathcal{H}_{Hg}$ nonempty for each $g \in [H\backslash G]$.

We say that $\mathcal{H}$ is a G/H-*graded left linear topology* on R if it satisfies the following conditions:

(1) $\mathcal{H}$ is rigid;

(2) $\mathcal{H}_{Hg}$ is a filter for all $g \in [H\backslash G]$, that is, $I \in \mathcal{H}_{Hg}$, $I' \in \mathcal{L}^{G/H}(R(g))$, $I \subseteq I' \Rightarrow I' \in \mathcal{H}_{Hg}$ and $I, I' \in \mathcal{H}_{Hg} \Rightarrow I \cap I' \in \mathcal{H}_{Hg}$;

(3) $I \in \mathcal{H}$, $r \in \mathrm{h}(R) \Rightarrow (I : r) \in \mathcal{H}$.

If $H = 1$ the above remarks show that our definition is equivalent to the definition of a G-graded linear topology given in [10, p. 490].

$\mathcal{H}$ is called a G/H-*graded left Gabriel topology* if it satisfies conditions (1), (2), (3) and

(4) If $I \in \mathcal{L}^{G/H}(R)$ and there is $I' \in \mathcal{H}$ such that $(I : r) \in \mathcal{H}$ for all $r \in \mathrm{h}(I')$ then $I \in \mathcal{H}$.

(Note that if $I \in \mathcal{L}_{Hg}^{G/H}(R)$ and $(I : r) \in \mathcal{H}$ for all $r \in \mathrm{h}(I')$ then $I' \in \mathcal{H}_{Hg}$.)

2.8. Next we define correspondences between left graded topologies on R and rigid subcategories of R-Gr.

(2.8.1) If $\mathcal{G}$ is a G-graded left linear (Gabriel) topology on R let

$$\mathcal{G}^{G/H} = \{J \in \mathcal{L}^{G/H}(R) \mid \text{ there is } I \in \mathcal{G} \text{ such that } I \subseteq J\}$$

(2.8.2) If $\mathcal{D}$ is a rigid closed subcategory of $(G/H, R)$-Gr let

$$\mathcal{H}_\mathcal{D} = \{I \in \mathcal{L}^{G/H}(R) \mid R(g)/I \in \mathcal{D} \text{ for some } g \in G\}$$

(2.8.3) If $\mathcal{H}$ is a G/H-graded left linear topology on R let

$$\mathcal{D}_\mathcal{H} = \{N \in (G/H, R)\text{-Gr} \mid \ell_R(n) \in \mathcal{H} \text{ for all } n \in \mathrm{h}(N)\}$$

The proof of the next result, where $\mathbf{U} = \mathbf{U}_{G/H}^{G/1}$ and $\mathbf{F} = \mathbf{F}_{G/H}^{G/1}$ is routine.

2.9. Proposition. a) *If $\mathcal{G}$ is a G-graded left linear (respectively Gabriel) topology on R then $\mathcal{G}^{G/H}$ is the smallest G/H-graded linear (respectively Gabriel) topology on R containing $\mathbf{U}(\mathcal{G})$;*

b) *The correspondence $\mathcal{D} \mapsto \mathcal{H}_\mathcal{D}$ and $\mathcal{H} \mapsto \mathcal{D}_\mathcal{H}$ are bijection between the rigid closed (respectively localizing) subcategories of $(G/H, R)$-Gr and the G/H-graded left linear (respectively Gabriel) topologies on R;*

c) *If $\mathcal{C}$ is the rigid closed subcategory of R-Gr corresponding to the topology $\mathcal{G}$ then $\mathcal{C}^{G/H} = \mathcal{D}_{\mathcal{G}^{G/H}}$ and $\mathcal{H}_{\mathcal{C}^{G/H}} = \mathcal{G}^{G/H}$;*

2.10. The Gabriel topology determined by an idempotent G-graded ideal. Let J be a idempotent G-graded two-sided ideal of R and

$$\mathcal{G} = \{I \in \mathcal{L}^{\mathrm{gr}}(R) \mid J \subseteq I\}.$$

Then $\mathcal{G}$ is a G-graded Gabriel topology on R and

$$\mathcal{G}^{G/H} = \{I \in \mathcal{L}^{G/H}(R) \mid J \subseteq I\}.$$

It is not difficult to verify (see also [19, Example 3, p.200]) That the following assertion are true for every $N \in (G/H, R)$-Gr:

(2.10.1) N is $\mathcal{G}^{G/H}$-torsion if and only if $JN = 0$, or equivalently, $J \otimes_R N = 0$;

(2.10.2) N is $\mathcal{G}^{G/H}$-closed if and only if the canonical morphism $N \to \mathrm{HOM}_{G/H,R}(J, N)$ is an isomorphism.

2.11. Let $K \leq H$ be subgroups of G. Since $\mathbf{U}_{G/H}^{G/1} = \mathbf{U}_{G/H}^{G/K} \circ \mathbf{U}_{G/K}^{G/1}$ and $\mathbf{F}_{G/H}^{G/1} = \mathbf{U}_{G/K}^{G/1} \circ \mathbf{F}_{G/H}^{G/K}$, the arguments of Propositions 2.2 and 2.3 show that we have the pair of adjoint functors $(\mathbf{U}_{G/H}^{G/K}, \mathbf{F}_{G/H}^{G/K})$ between $\mathcal{C}^{G/H}$ and $\mathcal{C}^{G/K}$ where $\mathcal{C}$ is a rigid closed subcategory of R-Gr. Using the fact that $\mathbf{U}$ is a separable functor, we easily deduce:

2.12. Proposition. *Let $\mathcal{C}$ be a rigid closed subcategory of R-Gr.*

a) *An object P of $\mathcal{C}^{G/K}$ is projective in $\mathcal{C}^{G/K}$ if and only if $\mathbf{U}(P)$ is projective in $\mathcal{C}^{G/H}$;*

b) *If a object M of $\mathcal{C}^{G/K}$ is a generator of $\mathcal{C}^{G/K}$ then $\mathbf{U}(M)$ is a generator of $\mathcal{C}^{G/H}$.*

2.13. Proposition. *Let $K \leq H$ be subgroups of G, $\mathcal{C}$ be a rigid closed subcategory of R-Gr, $M \in (G/K, R)$-Gr and $N \in (G/H, R)$-Gr.*

a) *If M is $\mathcal{C}^{G/K}$-torsionfree then $\mathbf{U}(M)$ is $\mathcal{C}^{G/H}$-torsionfree;*

b) *If N is $\mathcal{C}^{G/H}$-torsionfree then $\mathbf{F}(N)$ is $\mathcal{C}^{G/K}$-torsionfree;*

c) *If M is $\mathcal{C}^{G/K}$-injective and H/K is finite then $\mathbf{U}(M)$ is $\mathcal{C}^{G/H}$-injective;*

d) *If N is $\mathcal{C}^{G/H}$-injective then $\mathbf{F}(N)$ is $\mathcal{C}^{G/K}$-injective;*

e) $\mathbf{t}_{G/H}(\mathbf{U}_{G/H}^{G/K}(M)) = \mathbf{U}_{G/H}^{G/K}(\mathbf{t}_{G/K}(M))$.

Proof. a) We have that $\mathbf{t}_{\mathcal{C}^{G/K}}(M) = M$, so $\mathbf{U}(\mathbf{t}_{\mathcal{C}^{G/K}}(M)) = \mathbf{U}(M)$. But $\mathbf{U}(\mathbf{t}_{\mathcal{C}^{G/K}}(M))$ belongs to $\mathcal{C}^{G/H}$, hence $\mathbf{t}_{\mathcal{C}^{G/H}}(\mathbf{U}(M)) = \mathbf{U}(M)$.

b) Let $X \in \mathcal{C}^{G/K}$. Then $\mathrm{Hom}_{(G/K,R)\text{-Gr}}(X, \mathbf{F}(N)) \cong \mathrm{Hom}_{(G/H,R)\text{-Gr}}(\mathbf{U}(X), N) = 0$ since $\mathbf{U}(X) \in \mathcal{C}^{G/H}$. Consequently, $\mathbf{F}(N)$ is $\mathcal{C}^{G/K}$-torsion free.

c) and d) are easy consequences of the adjunction and of the fact that $\mathbf{U}$ and $\mathbf{F}$ preserves torsion objects.

e) Let $\mathcal{G}$ be the G-graded linear topology corresponding to $\mathcal{C}$. Then

$$\mathcal{G}^{G/H} = \{I \in \mathcal{L}^{G/H}(R) \mid \text{ there is } I \in \mathcal{G}^{G/K} \text{ such that } I \subseteq J\},$$

since we may take I to be a G-graded ideal. Now the argument of [10, Proposition 2.2] applies.

2.14. Adjoint functors between quotient categories. As in [16, Propositions 4.3 – 4.8] we may consider the following "relative situation". As our functor $\mathbf{F} = \mathbf{F}_{G/H}^{G/K}$ and $\mathbf{U} = \mathbf{U}_{G/H}^{G/K}$ satisfy all needed properties, the proof of the following statements are the same as in [16].

Let $\mathcal{A}^{G/K}$ and $\mathcal{C}^{G/K}$ be two rigid closed subcategories of $(G/K, R)$-Gr such that $\mathcal{C}^{G/K} \subseteq \mathcal{A}^{G/K}$. We have seen that if $\mathcal{C}^{G/K}$ is a localizing subcategory of $\mathcal{A}^{G/K}$ then $\mathcal{C}^{G/H}$ is a localizing subcategory of $\mathcal{A}^{G/H}$. Assume that this is the case. The functors $\mathbf{U}$ and $\mathbf{F}$ induce by restriction the functors

$$\mathcal{A}^{G/K} \underset{\mathbf{F}}{\overset{\mathbf{U}}{\rightleftarrows}} \mathcal{A}^{G/H} \quad \text{and} \quad \mathcal{C}^{G/K} \underset{\mathbf{F}}{\overset{\mathbf{U}}{\rightleftarrows}} \mathcal{C}^{G/H}.$$

Consider the canonical functors

$$\mathcal{A}^{G/K} \underset{\mathbf{i}^{G/K}}{\overset{\mathbf{a}^{G/K}}{\rightleftarrows}} \mathcal{A}^{G/K}/\mathcal{C}^{G/K} \quad \text{and} \quad \mathcal{A}^{G/H} \underset{\mathbf{i}^{G/H}}{\overset{\mathbf{a}^{G/H}}{\rightleftarrows}} \mathcal{A}^{G/H}/\mathcal{C}^{G/H} \ ,$$

and define the functors

$$\mathcal{A}^{G/K}/\mathcal{C}^{G/K} \underset{\overline{\mathbf{F}}}{\overset{\overline{\mathbf{U}}}{\rightleftarrows}} \mathcal{A}^{G/H}/\mathcal{C}^{G/H}$$

by $\overline{\mathbf{U}} = \mathbf{a}^{G/H} \circ \mathbf{U} \circ \mathbf{i}^{G/K}$ and $\overline{\mathbf{F}} = \mathbf{a}^{G/K} \circ \mathbf{F} \circ \mathbf{i}^{G/H}$.

These functors have the following properties:
(2.14.1) $\overline{\mathbf{F}}$ is a right adjoint of $\mathbf{U}$ and commute with direct sums;
(2.14.2) $\overline{\mathbf{U}} \circ \mathbf{a}^{G/K} = \mathbf{a}^{G/H} \circ \mathbf{U}$;
(2.14.3) $\overline{\mathbf{U}}$ and $\overline{\mathbf{F}}$ are exact;
(2.14.4) If $M \in \mathcal{A}^{G/K}/\mathcal{C}^{G/K}$ is projective (generator, small) then $\overline{\mathbf{U}}(M)$ is projective (generator, small) in $\mathcal{A}^{G/H}/\mathcal{C}^{G/H}$.

2.15. Rigid subcategories of $(G/H, R)$-Gr. If $\mathcal{C} = R$-Gr then clearly $\mathcal{C}^{G/H} = (G/H, R)$-Gr. Let M be a G-graded R-module and denote $\tilde{M} = \bigoplus_{g \in G} M(g)$. We shall consider several rigid full subcategories $\mathcal{C}$ of R-Gr associated with M and the corresponding rigid subcategories $\mathcal{C}^{G/H}$ of $(G/H, R)$-Gr. Again $\mathbf{U}$ means $\mathbf{U}_{G/H}^{G/1}$.

- If $\mathcal{C} = \boldsymbol{\sigma}^{\mathrm{gr}}[M] = \boldsymbol{\sigma}[\tilde{M}]$ then $\mathcal{C}^{G/H} = \boldsymbol{\sigma}^{G/H}[M] = \boldsymbol{\sigma}[\mathbf{U}(\tilde{M})]$;
- If $\mathcal{C} = \mathrm{Gen}^{\mathrm{gr}}[M] = \mathrm{Gen}[\tilde{M}]$ then $\mathcal{C}^{G/H} = \mathrm{Gen}^{G/H}[M] = \mathrm{Gen}[\mathbf{U}(\tilde{M})]$;
- If $\mathcal{C} = \mathrm{Pres}^{\mathrm{gr}}[M] = \mathrm{Pres}[\tilde{M}]$ then $\mathcal{C}^{G/H} = \mathrm{Pres}^{G/H}[M] = \mathrm{Pres}[\mathbf{U}(\tilde{M})]$.

By Proposition 2.12 it follows that M is projective in $\boldsymbol{\sigma}^{\mathrm{gr}}[M]$ if and only if it is projective in $\boldsymbol{\sigma}^{G/H}[M]$, or equivalently, M is a Σ-quasiprojective R-module. We also have that if $\tilde{M}$ is a generator of $\boldsymbol{\sigma}^{\mathrm{gr}}[M]$, then $\tilde{M}$ is a generator of $\boldsymbol{\sigma}^{G/H}[M]$.

Since for every $g \in G$, $\mathrm{HOM}_{G/H,R}(M, N)_{gH} = \mathrm{Hom}_{(G/HR)\text{-}\mathrm{Gr}}(M(g^{-1}, N)$, we obtain that

$$\mathrm{Im}\,\rho_N^{G/H} = \sum \{\mathrm{Im}\,f \mid f \in \mathrm{HOM}_{G/H,R}(M, N)\}$$

where $\rho_N^{G/H}$ is defined in 1.10. This implies that

$$\mathrm{Gen}^{G/H}[M] = \{N \in (G/H, R)\text{-}\mathrm{Gr} \mid \rho_N^{G/H} \text{ is an epimorphism }\}.$$

2.16. Σ-quasiprojective module. Assume in adition that the G-graded R-module M is Σ-quasiprojective. Let $\mathcal{T} = \mathcal{T}^{\mathrm{gr}}[M]$ be as in 1.6 the torsion class consisting of objects $M' \in R$-Gr satisfying $\mathrm{Hom}_{R\text{-}\mathrm{Gr}}(\tilde{M}, M') = 0$. Since

$$\mathrm{Hom}_{R\text{-}\mathrm{Gr}}(\tilde{M}, M') = \prod_{g \in G} \mathrm{Hom}_{R\text{-}\mathrm{Gr}}(M(g^{-1}), M') = \prod_{g \in G} \mathrm{Hom}_{R\text{-}\mathrm{Gr}}(M, M')_g,$$

we have that $M' \in \mathcal{T}$ if and only if $\mathrm{HOM}_R(M, M') = 0$. That implies that $\mathcal{T}$ is rigid, since $\mathrm{HOM}_R(M, M'(g)) = \mathrm{HOM}_R(M, M')(g)$ for all $g \in G$.

We claim that

$$\mathcal{T}^{G/H} = \{N \in \boldsymbol{\sigma}^{G/H}[M] \mid \mathrm{HOM}_{G/H,R}(M, N) = 0\}.$$

Indeed, by Proposition 2.3 a) and Corollary 2.4, it is enough to show that for any $M' \in \boldsymbol{\sigma}^{\mathrm{gr}}[M]$, $M' \in \mathcal{T}$ if and only if $\mathrm{HOM}_{G/H,R}(M, \mathbf{U}(M')) = 0$. But this follows immediately from 1.12. Moreover, since by 1.5

$$\mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\tilde{M}, N) = \prod_{g \in G} \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(M(g^{-1}), N)$$

$$= \prod_{g \in G} \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(M(g^{-1}, M)_{gH},$$

we have that $N \in \mathcal{T}^{G/H}$ if and only if $\mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\tilde{M}, N) = 0$, that is, $\mathcal{T}$ is the torsion theory on $\boldsymbol{\sigma}^{\mathrm{gr}}[M]$ determined by $\tilde{M}$. Finally, note that by [12, Theorem 3.7], $N \in \mathcal{T}^{G/H}$ if and only if $\mathrm{Hom}_R(M, N) = 0$.

These arguments also show that for the corresponding torsionfree class we have that for any $N \in \boldsymbol{\sigma}^{\mathrm{gr}}[M]$, N is $\tilde{M}$-faitful if and only if

$$X_M \neq 0 \text{ for every non zero subobject } X \text{ of } N$$

$$\Leftrightarrow \mathrm{Hom}_{(G/K,R)\text{-}\mathrm{Gr}}(\tilde{M}, X) \neq 0 \text{ for every non zero subobject } X \text{ of } N$$

$$\Leftrightarrow \mathrm{HOM}_{G/K,R}(M, X) \neq 0 \text{ for every non zero subobject } X \text{ of } N$$

$$\Leftrightarrow \mathrm{Hom}_R(M, X) \neq 0 \text{ for every non zero subobject } X \text{ of } N.$$

We shall also consider the following categories associated to M:

• If $\mathcal{C} = \mathcal{C}^{\mathrm{gr}}[M] = \boldsymbol{\sigma}^{\mathrm{gr}}[M]/\mathcal{T}^{\mathrm{gr}}$ (which can be identified with the full subcategory of $\boldsymbol{\sigma}^{\mathrm{gr}}[M]$ consisting of $\mathcal{T}$-closed objects), then $\mathcal{C}^{G/H} = \mathcal{C}^{G/H}[M] = \boldsymbol{\sigma}^{G/H}[M]/\mathcal{T}^{G/H}$ with the similar identification. By 2.14 it follows that we have the pair $(\overline{\mathbf{U}}, \overline{\mathbf{F}})$ of adjoint functors between $\mathcal{C}^{G/K}$ and $\mathcal{C}^{G/H}$.

• If $\mathcal{C} = \mathrm{GF}^{\mathrm{gr}}[M] = \mathrm{GF}[\tilde{M}]$ is the full subcategory of $\boldsymbol{\sigma}^{\mathrm{gr}}[M]$ consists of $\tilde{M}$-generated, $\mathcal{T}$-torsionfree objects, then again $\mathcal{C}^{G/H} = \mathrm{GF}^{G/H}[M] = \mathrm{GF}[\mathbf{U}(\tilde{M})]$. Again we have the adjoint pair $(\mathbf{U}, \mathbf{F})$ between $\mathrm{GF}^{G/K}[M]$ and $\mathrm{GF}^{G/H}[M]$.

2.17. The graded Gabriel topology on E. The G-graded Σ-quasiprojective module M determines a Gabriel topology on $S = \mathrm{End}_R(M)^{\mathrm{op}}$. If we denote by J^S the two-sided ideal of S consisting of the endomorphisms which factor through a finitely generated R-submodule of M, then by [7, Theorem 1.3], J^S is an idempotent ideal, $MJ^S = M$ and the associated Gabriel topology $\mathcal{G}^S$ consists of left ideals I of S satisfying $MI = M$.

Let $E = \mathrm{END}_R(M)$ and $J = E \cap J^S$. The next lemma and 2.10 show that

$$\mathcal{G}^{\mathrm{gr}} = \{I \in \mathcal{L}^{\mathrm{gr}}(R) \mid J \subseteq I\} = \{I \in \mathcal{L}^{\mathrm{gr}}(R) \mid MI = M\}$$

is a G-graded Gabriel topology on E. It also follows that if H is a subgroup of G, then

$$\mathcal{G}^{G/H} = \{I \in \mathcal{L}^{G/H}(R) \mid J \subseteq I\} = \{I \in \mathcal{L}^{G/H}(R) \mid MI = M\}$$

is a G/H-graded left Gabriel topology on E.

2.18. Lemma. *Let A be a subring of S containing E. Then the following statements hold.*

a) *J is an G-graded idempotent two-sided ideal of E;*

b) *$JA = E$;*

c) *$M \otimes_E A \cong M$.*

Proof. a) Let $\alpha = \sum_{i=1}^n \alpha_{g_i} \in J$ with $\alpha_{g_i} \in E$, and let M' be a finitely generated R-submodule of M such that $\operatorname{Im} \alpha \subseteq M'$. Replacing the generators of M' by their homogeneous components, we may assume that M' is a G-graded submodule of M. Let $m \in M$ be a homogeneous element. Then $\alpha_{g_i}(m) \in M$ are also homogeneous, and since $\alpha(m) = \sum_{i=1}^n \alpha_{g_i}(m) \in M'$, it follows that $\operatorname{Im} \alpha_{g_i} \subseteq M'$, that is, $\alpha_{g_i} \in J$, $1 \le i \le n$.

The remaining statements are proved as in [9, Theorem 2.1] and [7, Theorem 1.3], but working with homogeneous elements and grade preserving maps.

b) Let $\alpha \in J$ and $\beta \in A$. Thus we have the commutative diagram:

$$
\begin{array}{ccccc}
M & \xrightarrow{\ \ \alpha\ \ } & M & \xrightarrow{\ \beta\ } & M \\
& \searrow{\scriptstyle \alpha'} & \nearrow{\scriptstyle q} & & \\
& & M' & &
\end{array}
$$

with M' a G-graded finitely generated R-submodule of M. Since M' is finitely generated, $\beta \circ q \in \operatorname{Hom}_R(M', M) = \operatorname{HOM}_R(M', M)$ thus $\alpha\beta = \beta \circ \alpha = \beta \circ q \circ \alpha' \in \operatorname{HOM}_R(M, M) = E$.

c) We have the natural homomorphisms $\mu \colon M \otimes_E A \to M$, $\mu(m \otimes \beta) = m\beta = \beta(m)$ and $\nu \colon M \to M \otimes_E A$, $\nu(m) = m \otimes 1$. Then $(\mu \circ \nu)(m) = m$, and for any $m \otimes \beta \in M \otimes_E A$ we can find $m_i \in M$ and $\alpha_i \in J$, $1 \le i \le n$, such that $\sum_{i=1}^n m_i \alpha_i = m$ (for $MJ = M$). Consequently,

$$
(\nu \circ \mu)(m \otimes \beta) = \nu(m\beta) = m\beta \otimes 1 = \sum_{i=1}^n m_i \alpha_i \beta \otimes 1
$$

$$
= \sum_{i=1}^n m_i \otimes \alpha_i \beta = \sum_{i=1}^n m_i \alpha_i \otimes \beta = m \otimes \beta,
$$

so ν is the inverse of μ.

3. Σ-QUASIPROJECTIVE MODULES AND EQUIVALENCES

3.1. In this section we shall use the notations and assumptions of $2.15 - 2.18$. Recall that H is a subgroup of G, M is a G-graded Σ-quasiprojective R-module, $\tilde{M} = \bigoplus_{g \in G} M(g)$, $E = \operatorname{END}_R(M)^{\mathrm{op}}$, and J is the idempotent G-graded two-sided ideal of E consisting of endomorphisms which factor trough a (G-graded) finitely generated submodule of M. Then M determines the (hereditary) torsion class $\mathcal{T}^{G/H} \subseteq \sigma^{G/H}[M]$ as in 2.16, and the G/H-graded left Gabriel topology $\mathcal{G}^{G/H}$ on E as in 2.17.

Before proving our main results, we need several lemmas.

3.2. Lemma. *The functor* $\mathrm{HOM}_{G/H,R}(M,-)\colon \boldsymbol{\sigma}^{G/H}[M] \to (G/H,E)\text{-Gr}$ *is exact.*

Proof. For any $g \in G$, $M(g)$ is projective in $\boldsymbol{\sigma}^{G/H}[M]$ since it is a direct summand of $\tilde{M}$. Applying the exact functor $\mathrm{Hom}_{(G/H,R)\text{-Gr}}(M(g^{-1}),-)$ to an exact sequence $0 \to N' \to N \to N'' \to 0$ from $\boldsymbol{\sigma}^{G/H}[M]$, and using 1.9 we obtain the exact sequence

$$0 \to \mathrm{HOM}_{G/H,R}(M,N') \to \mathrm{HOM}_{G/H,R}(M,N) \to \mathrm{HOM}_{G/H,R}(M,N'') \to 0$$

in $(G/H,E)\text{-Gr}$, which proves the lemma.

3.3. Lemma. a) $M^n \in \mathrm{Stat}^{G/H}[M]$ *for all nonnegative integers n;*
   b) $\mathrm{Pres}^{G/H}[M] = \mathrm{Stat}^{G/H}[M]$;
   c) $\mathrm{Adst}^{G/H}[M] = \mathrm{Im}\,\mathrm{HOM}_{G/H,R}(M,-)$;
   d) *Let* $N \in (G/H,R)\text{-Gr}$, $X = \mathrm{HOM}_{G/H,R}(M,N) \in (G/H,E)\text{-Gr}$ *and denote*

$$(\rho_N^{G/H})_* = \mathrm{HOM}_{G/H,R}(M,\rho_N^{G/H}).$$

Then $\eta_X^{G/H}$ *and* $(\rho_N^{G/H})_*$ *are isomorphisms inverse to each other;*
   e) *Let* $X \in (G/H,E)\text{-Gr}$, $N = M \otimes_E X \in (G/H,E)\text{-Gr}$ *and denote*

$$(\eta_X^{G/H})_* = M \otimes_E \eta_X^{G/H}.$$

Then $\rho_N^{G/H}$ *and* $(\eta_X^{G/H})_*$ *are isomorphisms inverse to each other;*
   f) *If* $N \in \boldsymbol{\sigma}^{G/H}[M]$, *then* $\rho_N^{G/H}$ *has torsion kernel and cokernel.*

Proof. a) It is clear that we can apply Lemma 2.18 with $A = \mathrm{HOM}_{G/H,R}(M,M)$, so we have the natural isomorphisms $M \otimes_E \mathrm{HOM}_{G/H,R}(M,M^n) \cong M \otimes_E \mathrm{HOM}_{G/H,R}(M,M)^n$, that is, $M^n \in \mathrm{Stat}^{G/H}[M]$ for all n.

b) The fact that $M^{(\Lambda)} \in \mathrm{Stat}[M]$ for any set Λ follows by the argument used in [9, Theorem 2.1], observing that the homomorphism s considered there is actually graded of degree one in our case. This and Lemma 3.2 implies that $\mathrm{Pres}^{G/H}[M] \subseteq \mathrm{Stat}^{G/H}[M]$. Observe that the other inclusion always holds, since

$$\mathrm{Stat}[M] \subseteq \mathrm{Im}\,M \otimes_E \mathrm{HOM}_{G/H,R}(M,-) \subseteq \mathrm{Pres}^{G/H}[M].$$

c) follows from b) using [4, Theorem 1.6].

d) By c) we have that $X = \mathrm{HOM}_{G/H,R}(M,N) \in \mathrm{Adst}^{G/H}[M]$, hence $\eta_X^{G/H}$ is an isomorphism. From the adjunction we obtain $(\rho_N^{G/H})_* \circ \eta_X^{G/H} = 1_X$, hence $(\rho_N^{G/H})_*$ is the inverse of $\eta_X^{G/H}$.

e) is the dual of d).

f) For any $N \in \boldsymbol{\sigma}^{G/H}[M]$ we have the exact sequence

$$0 \to \mathrm{Ker}\,\rho_N^{G/H} \to M \otimes_E \mathrm{HOM}_{G/H,R}(M,N) \xrightarrow{\rho_N^{G/H}} N \to \mathrm{Coker}\,\rho_N^{G/H} \to 0.$$

in $(G/H,R)\text{-Gr}$. Applying the exact functor $\mathrm{HOM}_{G/H,R}(M,-)$, and having in the mind that $(\rho_N^{G/H})_*$ is an isomorphism, it follows that $\mathrm{HOM}_{G/H,R}(M,\mathrm{Ker}\,\rho_N^{G/H}) = 0$, that is $\mathrm{Ker}\,\rho_N^{G/H}$ belongs to $\mathcal{T}^{G/H}$. For the cokernel, it is enough to observe that $\mathrm{Coker}\,\rho_N^{G/H} = N/\mathrm{Im}\,\rho_N^{G/H}$, $\mathrm{Im}\,\rho_N^{G/H} = N_{\tilde{M}}$, and the torsion theory class $\mathcal{T}^{G/H}[M]$ is generated by the objects $N/N_{\tilde{M}}$ with $N \in \boldsymbol{\sigma}^{G/H}[M]$.

3.4. Lemma. *If $g \in G$ and I is a G/H-graded left ideal of $E(g)$, then the induced homomorphism $M \otimes_E I \to M(g)$ has torsion kernel in $\boldsymbol{\sigma}^{G/H}[M]$.*

Proof. First assume that I is G-graded. We claim that

$$\eta_I^{\mathrm{gr}} : I \to \mathrm{HOM}_R(M, M \otimes_E I)$$

is an isomorphism. Indeed we have the commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & I & \longrightarrow & E(g) \\
 & & \downarrow{\scriptstyle \eta_I^{\mathrm{gr}}} & & \downarrow{\scriptstyle \eta_{E(g)}^{\mathrm{gr}}} \\
 & & \mathrm{HOM}_R(M, M \otimes_E I) & \longrightarrow & \mathrm{HOM}_R(M, M \otimes_E E(g))
\end{array}
$$

with first row exact, and since $\mathrm{HOM}_R(M, M \otimes_E E(g)) \cong \mathrm{HOM}_R(M, M(g)) \cong E(g)$, it follows that η_I^{gr} is a monomorphism.

Since I is finitely generated, it follows that there is an epimorphism $\bigoplus_{h \in G_f} E(h)^{k_h} \to I$, for some finite subset G_f of G and some natural numbers k_h, $h \in G_f$. Since the functor $\mathrm{HOM}_R(M, M \otimes_E -)$ is exact, we obtain the commutative diagram

$$
\begin{array}{ccccc}
\bigoplus_{h \in G_f} E(h)^{k_h} & \longrightarrow & I & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \eta_I^{\mathrm{gr}}} & & \\
\mathrm{HOM}_R(M, M \otimes_E (\bigoplus_{h \in G_f} E(h)^{k_h})) & \longrightarrow & \mathrm{HOM}_R(M, M \otimes_E I) & \longrightarrow & 0
\end{array}
$$

with exact rows. Moreover,

$$\mathrm{HOM}_R(M, M \otimes_E (\bigoplus_{h \in G_f} E(h)^{k_h})) \cong \bigoplus_{h \in G_f} (\mathrm{HOM}_R(M, (M \otimes_E E)(h)))^{k_h} \cong \bigoplus_{h \in G_f} E(h)^{k_h},$$

so η_I^{gr} is an epimorphism and our claim follows.

Now let $K = \mathrm{Ker}(M \otimes_E I \to M(g))$ for I a G-graded finitely generated ideal of $E(g)$. The first diagram shows that $\mathrm{HOM}_R(M, K) = 0$, that means, $K \in \mathcal{T}^{\mathrm{gr}}$. Thus $\mathbf{U}_{G/H}^{G/1}(K) \in \mathcal{T}^{G/H}$ for every $H \leq G$.

Finally, let I be a G/H-graded left ideal of $E(g)$ as in the hypothesis of the lemma. We may write $I = \sum_{\lambda \in \Lambda} I_\lambda$, where $\{I_\lambda \mid \lambda \in \Lambda\}$ is the set of all (ungraded) finitely generated subideals of I. For $\lambda \in \Lambda$, the set of G-homogeneous components of a finite set of generators of I_λ is finite too, and this set generates a G-graded left ideal, say I'_λ, containing I_λ. This means that the set of finitely generated G-graded left subideals of I is cofinal in $\{I_\lambda \mid \lambda \in \Lambda\}$, so $I = \sum_{\lambda \in \Lambda} I'_\lambda$. If K_λ denotes the kernel of the homomorphism $M \otimes_E I'_\lambda \to M(g)$, $\lambda \in \Lambda$, then we already have seen that $K_\lambda \in \mathcal{T}^{G/H}$. Since direct limits are exact, we have the isomorphism $\mathrm{Ker}(M \otimes_E I \to M(g)) \cong \sum_{\lambda \in \Lambda} K_\lambda$, hence $\mathrm{Ker}(M \otimes_E I \to M(g)) \in \mathcal{T}^{G/H}$.

3.5. Lemma. *Let $X \to Y$ be a monomorphism in $(G/H, E)$-Gr. Then the induced homomorphism $M \otimes_E X \to M \otimes_E Y$ has torsion kernel in $\boldsymbol{\sigma}^{G/H}[M]$.*

Proof. By standard arguments as in [18, Propositions 10.4 and 10.6] we see that this lemma is actually equivalent to the previous one and to the statement that $\mathrm{HOM}_{G/H,R}(M, Q)$ is an injective object of $(G/H, R)$-Gr, where Q is an arbitrary injective cogenerator of torsion theory $(\mathcal{T}^{G/H}, \mathcal{F}^{G/H})$.

3.6. Lemma. *The G-graded (R, E)-bimodules $M \otimes_E J$ and M are isomorphic.*

Proof. The image of the canonical homomorphism $\mu \colon J \otimes_E M \to M$ is MJ and $MJ = M$, so μ is surjective. But μ is induced by the monomorphism $0 \to J \to E$, hence its kernel K is torsion. The projectivity of M in $\boldsymbol{\sigma}^{G/H}[M]$ implies that the short exact sequence $0 \to K \to M \otimes_E J \to M \to 0$ splits, and K, as an epimorphic image of $M \otimes_E J \in \mathrm{Gen}^{G/H}[M]$, belongs to $\mathrm{Gen}^{G/H}[M]$. Consequently, $K = 0$ and μ is an isomorphism.

3.7. Lemma. *If X is a G/H-graded E-module, then the following are equivalent:*
  (i) *X is $\mathcal{G}^{G/H}$-torsion;*
  (ii) *$J \otimes_E X = 0$;*
  (iii) *$M \otimes_E X = 0$.*

Proof. (i)$\Rightarrow$(ii) Let $\alpha \otimes x \in J \otimes_E X$. Since J is idempotent we can find $\beta_i, \gamma_i \in J$ such that $\alpha = \sum_{i=1}^n \beta_i \gamma_i$. Then $\alpha \otimes x = \sum_{i=1}^n \beta_i \gamma_i \otimes x = \sum_{i=1}^n \beta_i \otimes \gamma_i x = 0$, hence $J \otimes_E X = 0$.

(ii)$\Rightarrow$(iii) By Lemma 3.6 we have that $M \otimes_E X \cong M \otimes_E J \otimes_E X = M \otimes_E 0 = 0$.

(iii)$\Rightarrow$(i) First, we shall show that $M \otimes_E X' = 0$ for all subobjects X' of X in $(G/H, R)$-Gr. Indeed, if X' is such a subobject, then

$$M \otimes_E X' = \mathrm{Ker}(M \otimes_E X' \to 0) = \mathrm{Ker}(M \otimes_E X' \to M \otimes_E X)$$

belongs to $\mathcal{T}^{G/H}$, and clearly $M \otimes_E X' \in \mathrm{Gen}^{G/H}[M]$, hence $M \otimes_E X' = 0$.

Now let $x \in X_{g^{-1}H}$. Since $\ell_R(x) \in \mathcal{L}_{Hg}^{G/H}(R)$, we obtain the short exact sequence

$$0 \to \ell_R(x) \to E(g) \to Ex \to 0.$$

Since Ex is a subobject of X, we have $M \otimes_E Ex = 0$, hence the above exact sequence induce an epimorphism $M \otimes_E \ell_R(x) \to M(g)$. On the other hand, the image of the map $M \otimes_E \ell_R(x) \to M(g)$ is $M\ell_R(x)$, so $M = M\ell_R(x)$ and $\ell_R(x) \in \mathcal{G}^{G/H}$.

3.8. Lemma. *The functor $\mathrm{HOM}_{G/H,R}(M, -) \colon (G/H, R)$-Gr $\to (G/H, E)$-Gr factors through the inclusion $(G/H, E, \mathcal{G}^{G/H})$-Gr $\to (G/H, E)$-Gr.*

Proof. Let $N \in (G/H, R)$-Gr. By Lemmas 1.11 and 3.6 we have the canonical isomorphisms

$$\mathrm{HOM}_{G/H,E}(J, \mathrm{HOM}_{G/H,R}(M, N)) \cong \mathrm{HOM}_{G/H,R}(M \otimes_E J, N) \cong \mathrm{HOM}_{G/H,R}(M, N),$$

and by 2.10 this means that $\mathrm{HOM}_{G/H,R}(M, N)$ is $\mathcal{G}^{G/H}$-closed.

3.9. Theorem. *The functor* $\mathrm{HOM}_{G/H,R}(M,-)\colon (G/H,R)\text{-}\mathrm{Gr} \to (G/H,E)\text{-}\mathrm{Gr}$ *restricts to the following equivalences of categories:*

a) $\mathrm{Pres}^{G/H}[M] \to (G/H,E,\mathcal{G}^{G/H})\text{-}\mathrm{Gr}$ *with inverse* $M \otimes_E -$;

b) $\mathrm{GF}^{G/H}[M] \to (G/H,E,\mathcal{G}^{G/H})\text{-}\mathrm{Gr}$ *with inverse* $\overline{M \otimes_E -}$;

c) $\mathcal{C}^{G/H}[M] \to (G/H,E,\mathcal{G}^{G/H})\text{-}\mathrm{Gr}$ *with inverse* $\mathbf{a}^{G/H} \circ (M \otimes_E -)$.

Proof. a) We have seen that the functors $\mathrm{HOM}_{G/H,R}(M,-)$ and $M \otimes_E -$ are well defined between $\mathrm{Pres}^{G/H}[M]$ and $(G/H,E,\mathcal{G}^{G/H})$.

Let $N \in \mathrm{Pres}^{G/H}[M]$. Then $\rho_N^{G/H}\colon M \otimes_E \mathrm{HOM}_{G/H,R}(M,N) \to N$ is an isomorphism by Lemma 3.3 b). Let $X \in (G/H,E,\mathcal{G}^{G/H})\text{-}\mathrm{Gr}$ and put $K = \mathrm{Ker}\,\eta_X^{G/H}$, $C = \mathrm{Coker}\,\eta_X^{G/H}$. Tensoring with M the exact sequence

$$0 \to K \to X \xrightarrow{\eta_X^{G/H}} \mathrm{HOM}_{G/H,R}(M, M \otimes_E X) \to C \to 0$$

we obtain the exact sequence

$$M \otimes_E X \xrightarrow{(\eta_X^{G/H})_*} M \otimes_E \mathrm{HOM}_{G/H,R}(M, M \otimes_E X) \to M \otimes_E C \to 0.$$

By Lemma 3.3 d), $(\eta_X^{G/H})_*$ is an isomorphism, hence $M \otimes_E C = 0$, so by Lemma 3.7, C is $\mathcal{G}^{G/H}$ torsion. Moreover, the induced homomorphism $M \otimes_E K \to M \otimes_E X$ is zero, hence $M \otimes_E K = \mathrm{Ker}(M \otimes_E K \to M \otimes_E X)$ belongs to $\mathcal{T}^{G/H}$. On other hand, it is clear that $M \otimes_E K \in \mathrm{Gen}^{G/H}[M]$, so $M \otimes_E K = 0$, and again by Lemma 3.7, K is $\mathcal{G}^{G/H}$ torsion.

b) We can use the same argument as [7, Theorem 1.3 (ii)], replacing $\mathrm{Hom}_R(-,-)$ with $\mathrm{HOM}_{G/H,R}(-,-)$.

c) We have the diagram of categories and functors

$$
\begin{array}{ccc}
 & \boldsymbol{\sigma}^{G/H}[M] & \\
 {}^{\mathbf{j}}\nearrow\!\!\!\swarrow & & \searrow\!\!\!\nwarrow{}^{\mathbf{a}} \\
 & {}_{\mathbf{i}} & \\
\mathrm{Pres}^{G/H}[M] & \underset{M\otimes_E \mathrm{HOM}_{G/H,R}(M,-)}{\overset{\mathbf{a}\circ\mathbf{j}}{\rightleftarrows}} & \mathcal{C}^{G/H}[M]
\end{array}
$$

where $\mathbf{i} = \mathbf{i}^{G/H}$, $\mathbf{j} = \mathbf{j}^{G/H}$ are the corresponding inclusion functors and $\mathbf{a} = \mathbf{a}^{G/H}$ is the canonical adjoint of $\mathbf{i}$. We claim that on the row we have equivalences inverse to each other. In order to prove that, observe first that the functor $M \otimes_E \mathrm{HOM}_{G/H,R}(M,-)\colon \boldsymbol{\sigma}^{G/H}[M] \to \mathrm{Pres}^{G/H}[M]$ is the right adjoint of the inclusion $\mathbf{j}$, that is $\mathrm{Pres}^{G/H}[M]$ is a reflective subcategory of $\boldsymbol{\sigma}^{G/H}[M]$. Indeed, Lemma 3.3 gives the natural isomorphisms

$$\mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(N, M \otimes_E \mathrm{HOM}_{G/H,R}(M,L))$$
$$\cong \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(M \otimes_E \mathrm{HOM}_{G/H,R}(M,N), M \otimes_E \mathrm{HOM}_{G/H,R}(M,L))$$
$$\cong \mathrm{Hom}_{(G/H,E)\text{-}\mathrm{Gr}}(\mathrm{HOM}_{G/H,R}(M,N), \mathrm{HOM}_{G/H,R}(M,L)),$$
$$\mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(N, L) \cong \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\mathrm{HOM}_{G/H,R}(M,N), L)$$
$$\cong \mathrm{Hom}_{(G/H,E)\text{-}\mathrm{Gr}}(\mathrm{HOM}_{G/H,R}(M,N), \mathrm{HOM}_{G/H,R}(M,L))$$

for every $N \in \mathrm{Pres}^{G/H}[M]$ and every $L \in \boldsymbol{\sigma}^{G/H}[M]$. Moreover, the reflector $M \otimes_E$ $\mathrm{HOM}_{G/H,R}(M, -)$ is right exact between $\boldsymbol{\sigma}^{G/H}[M]$ and itself. Since the inclusion $\mathbf{j}$ is also right exact, this implies that the reflector is exact between $\boldsymbol{\sigma}^{G/H}[M]$ and $\mathrm{Pres}^{G/H}[M]$. Thus, by the duals of [13, 5.1, 5.2, 5.3, Chapter V], we have that

- $\mathrm{Pres}^{G/H}[M]$ is abelian;
- A morphism in $\mathrm{Pres}^{G/H}[M]$ is epimorphism in $\mathrm{Pres}^{G/H}[M]$ if and only if it is epimorphism in $\boldsymbol{\sigma}^{G/H}[M]$;
- A morphism in $\mathrm{Pres}^{G/H}[M]$ is monomorphism in $\mathrm{Pres}^{G/H}[M]$ if and only if the functor $M \otimes_E \mathrm{HOM}_{G/H,R}(M, -)$ maps its kernel to zero in $\boldsymbol{\sigma}^{G/H}[M]$.

Note that for $N \in \boldsymbol{\sigma}^{G/H}[M]$, the equality $M \otimes_E \mathrm{HOM}_{G/H,R}(M, N) = 0$ implies that $\mathrm{HOM}_{G/H,R}(M, N)$ is $\mathcal{G}^{G/H}$-torsion and, since it is always torsionfree (even closed), we have that $\mathrm{HOM}_{G/H,R}(M, N) = 0$. Since the converse is obvious, $M \otimes_E \mathrm{HOM}_{G/H,R}(M, N) = 0$ if and only if $N \in \mathcal{T}^{G/H}$. In addition, the functor $M \otimes_E \mathrm{HOM}_{G/H,R}(M, -) \colon \boldsymbol{\sigma}^{G/H}[M] \to \mathrm{Pres}^{G/H}[M]$ is left exact, since it is a right adjoint of $\mathbf{j}$, and also right exact by the above argument.

The inclusion functor $\mathbf{j}$ is obviously fully-faithful, so by [18, 13.11, Chapter I] it follows that $\mathrm{Pres}^{G/H}[M]$ is equivalent to the category of fractions of $\boldsymbol{\sigma}^{G/H}[M]$ relative to the system

$$\Sigma = \{f \in \mathrm{Hom}\,\boldsymbol{\sigma}^{G/H}[M] \mid M \otimes_E \mathrm{HOM}_{G/H,R}(M, f) \text{ is invertible in } \mathrm{Pres}^{G/H}[M]\}.$$

To prove our claim, it is now enough to show that

$$\Sigma = \{f \in \mathrm{Hom}\,\boldsymbol{\sigma}^{G/H}[M] \mid \mathrm{Ker}\,f, \mathrm{Coker}\,f \in \mathcal{T}^{G/H}\}.$$

Indeed, let $f \in \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(N, L)$ with $N, L \in \boldsymbol{\sigma}^{G/H}[M]$ and $\mathrm{Ker}\,f, \mathrm{Coker}\,f \in \mathcal{T}^{G/H}$. The exact sequence

$$0 \to \mathrm{Ker}\,f \to N \xrightarrow{f} L \to \mathrm{Coker}\,f \to 0$$

induce the exact sequence in $\mathrm{Pres}^{G/H}[M]$

$$0 \to M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Ker}\,f) \to M \otimes_E \mathrm{HOM}_{G/H,R}(M, N) \xrightarrow{f_{**}}$$
$$M \otimes_E \mathrm{HOM}_{G/H,R}(M, L) \to M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Coker}\,f) \to 0,$$

where f_{**} denotes the homomorphism $M \otimes_E \mathrm{HOM}_{G/H,R}(M, f)$. Since

$$M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Ker}\,f) = M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Coker}\,f) = 0$$

in $\boldsymbol{\sigma}^{G/H}[M]$, this equality holds in $\mathrm{Pres}^{G/H}[M]$ too, and f_{**} is an isomorphism in $\mathrm{Pres}^{G/H}[M]$.

Conversely, suppose that f_{**} is an isomorphism in $\mathrm{Pres}^{G/H}[M]$. Since this is a full subcategory of $\boldsymbol{\sigma}^{G/H}[M]$ it follows that f_{**} is isomorphism in $\boldsymbol{\sigma}^{G/H}[M]$ too. The exact sequence in $\boldsymbol{\sigma}^{G/H}[M]$

$$M \otimes_E \mathrm{HOM}_{G/H,R}(M, N) \xrightarrow{f_{**}} M \otimes_E \mathrm{HOM}_{G/H,R}(M, L) \to$$
$$M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Coker}\,f) \to 0$$

shows that $M \otimes_E \mathrm{HOM}_{G/H,R}(M, \mathrm{Coker}\, f) = 0$ in $\boldsymbol{\sigma}^{G/H}[M]$, that is, $\mathrm{Coker}\, f \in \mathcal{T}^{G/H}$. The Ker-Coker lemma for the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_E \mathrm{HOM}_{G/H,R}(M, N) & \xrightarrow{f_{**}} & M \otimes_E \mathrm{HOM}_{G/H,R}(M, L) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \rho_N^{G/H}} & & \downarrow{\scriptstyle \rho_L^{G/H}} \\
0 & \longrightarrow & \mathrm{Ker}\, f & \longrightarrow & N & \xrightarrow{\quad f \quad} & L
\end{array}
$$

implies that there is an exact sequence

$$
\mathrm{Ker}\, \rho_L^{G/H} \to \mathrm{Ker}\, f \to \mathrm{Coker}\, \rho_N^{G/H}.
$$

But by Lemma 3.3 f), $\mathrm{Ker}\, \rho_L^{G/H}$ and $\mathrm{Coker}\, \rho_N^{G/H}$ belong to $\mathcal{T}^{G/H}$, hence $\mathrm{Ker}\, f \in \mathcal{T}^{G/H}$, and this completes the proof of our claim.

Composing this equivalences with those obtained in a), and using the fact that $(\rho_N^{G/H})_*$ is an isomorphism for all $N \in \boldsymbol{\sigma}^{G/H}[M]$, we obtain the desired equivalence between $\mathcal{C}^{G/H}[M]$ and $(G/H, E, \mathcal{G}^{G/H})\text{-Gr}$.

3.10. Note that the above diagram of categories and functors in not necessarily commutative, since the functors $\mathbf{i} \circ \mathbf{a} \circ \mathbf{j}$ and $\mathbf{j}$ are not necessarily isomorphic.

3.11. Corollary. *Let $K \le H \le G$ be two subgroups, and let $\mathcal{G}^{G/K}$ and $\mathcal{G}^{G/K}$ the G/K (respectively G/H)-graded left topologies considered in Theorem 3.9. Then there are the following commutative diagram of categories and functors:*

a)

$$
\begin{array}{ccc}
\mathrm{Pres}^{G/K}[M] & \underset{M \otimes_E -}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/K, E, \mathcal{G}^{G/K})\text{-Gr} \\
{\scriptstyle \mathbf{F}_{G/H}^{G/K}} \big\Vert {\scriptstyle \mathbf{U}_{G/H}^{G/K}} & & {\scriptstyle \overline{\mathbf{F}}_{G/H}^{G/K}} \big\Vert {\scriptstyle \overline{\mathbf{U}}_{G/H}^{G/K}} \\
\mathrm{Pres}^{G/H}[M] & \underset{M \otimes_E -}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/H, E, \mathcal{G}^{G/H})\text{-Gr}
\end{array}
$$

b)

$$
\begin{array}{ccc}
\mathrm{GF}^{G/K}[M] & \underset{\overline{M \otimes_E -}}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/K, E, \mathcal{G}^{G/K})\text{-Gr} \\
{\scriptstyle \mathbf{F}_{G/H}^{G/K}} \big\Vert {\scriptstyle \mathbf{U}_{G/H}^{G/K}} & & {\scriptstyle \overline{\mathbf{F}}_{G/H}^{G/K}} \big\Vert {\scriptstyle \overline{\mathbf{U}}_{G/H}^{G/K}} \\
\mathrm{GF}^{G/H}[M] & \underset{\overline{M \otimes_E -}}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/H, E, \mathcal{G}^{G/H})\text{-Gr}
\end{array}
$$

c)

$$
\begin{array}{ccc}
\mathcal{C}^{G/K}[M] & \underset{\mathbf{a}^{G/K} \circ (M \otimes_E -)}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/K, E, \mathcal{G}^{G/K})\text{-Gr} \\
{\scriptstyle \overline{\mathbf{F}}_{G/H}^{G/K}} \big\Vert {\scriptstyle \overline{\mathbf{U}}_{G/H}^{G/K}} & & {\scriptstyle \overline{\mathbf{F}}_{G/H}^{G/K}} \big\Vert {\scriptstyle \overline{\mathbf{U}}_{G/H}^{G/K}} \\
\mathcal{C}^{G/H}[M] & \underset{\mathbf{a}^{G/H} \circ (M \otimes_E -)}{\overset{\mathrm{HOM}_{G/K,R}(M,-)}{\rightleftarrows}} & (G/H, E, \mathcal{G}^{G/H})\text{-Gr}
\end{array}
$$

Proof. a) Let $N \in \mathrm{Pres}^{G/K}[M]$. We have the short exact sequence

$$0 \to \mathrm{HOM}_{G/K,R}(M,N) \to \mathrm{HOM}_{G/H,R}(M,N) \to C \to 0$$

in $(G/H,E)$-Gr for $C = \mathrm{Coker}\,\mathrm{HOM}_{G/K,R}(M,N) \to \mathrm{HOM}_{G/H,R}(M,N)$. Tensoring with M, and having in the mind that by Theorem 3.9, $M \otimes_E \mathrm{HOM}_{G/K,R}(M,N) \cong N \cong M \otimes_E \mathrm{HOM}_{G/H,R}(M,N)$, we obtain that $M \otimes_E C = 0$, hence C is $\mathcal{G}^{G/H}$-torsion. Consequently, in $(G/H,E,\mathcal{G}^{G/H})$-Gr we have the isomorphism

$$\overline{\mathbf{U}_{G/H}^{G/K}}(\mathrm{HOM}_{G/K,R}(M,N)) \cong \mathrm{HOM}_{G/H,R}(M,\mathbf{U}_{G/H}^{G/K}(N)).$$

Since the rows are equivalences, $\mathbf{F}_{G/H}^{G/K}$ is the right adjoint of $\mathbf{U}_{G/H}^{G/K}$ and $\overline{\mathbf{F}_{G/H}^{G/K}}$ is the right adjoint of $\overline{\mathbf{U}_{G/H}^{G/K}}$, it is clear that the diagram commutes.

b) By Theorem 3.9 every object of $\mathrm{GF}^{G/K}[M]$ can be regarded as $\bar{N}$ for a suitable $N \in \mathrm{Pres}^{G/K}[M]$. Since $\mathrm{HOM}_{G/K,R}(M,N) \cong \mathrm{HOM}_{G/K,R}(M,\bar{N})$, we have by a) the isomorphisms

$$\overline{\mathbf{U}_{G/H}^{G/K}}(\mathrm{HOM}_{G/K,R}(M,\bar{N})) \cong \overline{\mathbf{U}_{G/H}^{G/K}}(\mathrm{HOM}_{G/K,R}(M,N)) \cong \mathrm{HOM}_{G/H,R}(M,\mathbf{U}_{G/H}^{G/K}(N))$$

in $(G/H,E,\mathcal{G}^{G/H})$-Gr.

On the other hand, by 2.13 d), $\mathbf{t}_{G/H}(\mathbf{U}_{G/H}^{G/K}(N)) = \mathbf{U}_{G/H}^{G/K}(\mathbf{t}_{G/H}(N))$ so

$$\mathrm{HOM}_{G/H,R}(M,\mathbf{U}_{G/H}^{G/K}(\bar{N})) \cong \mathrm{HOM}_{G/H,R}(M,\mathbf{U}_{G/H}^{G/K}(N)).$$

Again, this is enough for the commutativity of the diagram.

c) Let $X \in (G/H,E,\mathcal{G}^{G/H})$-Gr. By a) we have the isomorphism $M \otimes_E \overline{\mathbf{U}_{G/H}^{G/K}}(X) \cong \mathbf{U}_{G/H}^{G/K}(M \otimes_E X)$, since both members belong to $\mathrm{Pres}^{G/K}[M]$. On the other hand, using (2.14.2), we obtain

$$(\overline{\mathbf{U}_{G/H}^{G/K}} \circ \mathbf{a}^{G/K})(M \otimes_E X) \cong (\mathbf{a}^{G/H} \circ \mathbf{U}_{G/H}^{G/K})(M \otimes_E X) \cong \mathbf{a}^{G/H}(M \otimes_E \mathbf{U}_{G/H}^{G/K}(X)),$$

so the diagram is commutative.

3.12. Theorem. *Let* $S = \mathrm{End}_R(M)$, $\mathcal{G}^S = \{I \in \mathcal{L}(S) \mid J^S \subseteq I\}$ *as in 2.17, and let* $\boldsymbol{\varphi}^*$ *and* $\boldsymbol{\varphi}_*$ *denote the restriction, respectively the extension of scalars induced by the inclusion* $\varphi \colon E \to S$. *Then*

a) *S is isomorphic to the ring of the quotients of E with respect the topology $\mathcal{G}^{G/G}$;*

b) *There is the following commutative diagram of categories and functors*

$$
\begin{array}{ccc}
\mathrm{Pres}^{G/G}[M] & \underset{M \otimes_E -}{\overset{\mathrm{Hom}_R(M,-)}{\rightleftarrows}} & (E,\mathcal{G}^{G/G})\text{-Mod} \\
\| & & \overline{\varphi^*} \Big\Updownarrow \overline{\varphi_*} \\
\mathrm{Pres}[M] & \underset{M \otimes_S -}{\overset{\mathrm{Hom}_R(M,-)}{\rightleftarrows}} & (S,\mathcal{G})\text{-Mod},
\end{array}
$$

where $\overline{\varphi_*}$ and $\overline{\varphi^*}$ are the induced functor between the quotient categories.

Proof. a) Regarding S as an object of E-Mod, we have the short exact sequence of E-modules

$$0 \to E \xrightarrow{\varphi} S \to \operatorname{Coker} \varphi \to 0,$$

which induces the exact sequence in R-Mod

$$M \otimes_E E \to M \otimes_E S \to M \otimes_E \operatorname{Coker} \varphi \to 0.$$

By 2.18, we have that $M \otimes_E E \cong M \cong M \otimes_E S$, so $M \otimes_E \operatorname{Coker} \varphi = 0$, hence $\operatorname{Coker} \varphi$ is $\mathcal{G}^{G/G}$-torsion. Moreover, $S = \operatorname{Hom}_R(M, M)$ is $\mathcal{G}^{G/G}$-closed by Lemma 3.8, hence [3, Lemma 2.6] implies our statement.

b) The fact that the second row is an equivalence is just [7, Theorem 1.3 a)]. If $N \in \operatorname{Pres}[M]$, then we have the natural S-homomorphisms

$$\phi_N \colon \operatorname{Hom}_R(M, N) \to S \otimes_E \operatorname{Hom}_R(M, N), \quad \phi(f) = 1 \otimes f$$
$$\psi_N \colon S \otimes_E \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N), \quad \psi_N(\alpha \otimes f) = \alpha f = f \circ \alpha.$$

We have that $\psi_N \circ \phi_N = 1_{\operatorname{Hom}_R(M,N)}$, so ϕ_N is an monomorphism. Moreover, $\operatorname{Coker} \phi_N$ is $\mathcal{G}^S$-torsion. Indeed the argument given in Corollary 3.11 shows that $M \otimes_S C = 0$, and by an argument similar to that in Lemma 3.7, this implies that $C \in \mathcal{G}$. Thus ϕ_N is an isomorphism in $(S, \mathcal{G}^S)$-Mod.

We also have that $\operatorname{Hom}_R(M, N)$ is $\mathcal{G}$-closed, hence it is isomorphic in $(S, \mathcal{G}^S)$-Mod to its module of quotients. By the definition of the functor $\overline{\varphi^*}$, $\overline{\varphi^*}(\operatorname{Hom}_R(M, N)) \cong \operatorname{Hom}_R(M, N)$, and the diagram commutes.

4. Applications and examples

4.1. We continue to use the notations and assumptions of 3.1. Observe that M is finitely generated if and only if $J = E$, and in this case we have that $E = S$. This situation was discussed in [11, Theorem 3.12]. In particular if M is a progenerator of R-Mod we obtain a graded Morita equivalence between R and S.

4.2. If M is a simple object of R-Gr, then $E = S$ and M is projective generator of $\boldsymbol{\sigma}[M]$, hence $\boldsymbol{\sigma}[M]$ is equivalent with E-Mod. This is the main result of [5], the so called "direct Clifford theorem". This was generalized in [10, Corollary 2.11] to the case when M is semisimple in R-Gr, but with S instead of E.

4.3. Assume that E is strongly graded. Then we know that J_1 is a G-invariant two-sided ideal of E_1, and we claim that it is also idempotent.

Indeed, if $\alpha \in J_1$, then $\alpha = \sum_{i=1}^n \beta_i \gamma_i$ for some $\beta_i, \gamma_i \in J$, and we may clearly assume that β_i, γ_i are homogeneous. Fix i and assume that $\beta_i \in E_g$, so $\gamma_i \in E_{g^{-1}}$. Since $E_{g^{-1}} E_g = E_1$, we can find $\epsilon'_j, \in E_{g^{-1}}$, $\epsilon_J \in E_g$ $1 \le j \le M$ such that $\sum_{j=1}^m \epsilon'_j \epsilon_j = 1$. It follows that $\beta_i \epsilon'_j, \epsilon_j \gamma_i \in E_1$ and $\beta_i \gamma_i = \sum_{j=1}^m (\beta_i \epsilon'_j)(\gamma_i \epsilon_j)$, and our claim follows immediately.

Similarly, J_H is a idempotent ideal of E_H for every subgroup H of G. The functors $E \otimes_{E_H} -$ and $(-)_H$ give an equivalence between $(G/H, E)$-Gr and E_H-Mod, so for a G/H-graded module X we have a natural isomorphism $X \cong E \otimes_{E_H} X$, and similar statements hold for right modules. In particular we have that

$$M \otimes_E X \cong M \otimes_{E_H} X.$$

Moreover, the natural homomorphism $X \to \mathrm{HOM}_{G/H,E}(J, X)$ of G/H-graded E-modules comes, via the functor $E \otimes_{E_H} -$, from a unique homomorphism $X_H \to \mathrm{Hom}_{E_H}(J_H, X_H)$ of E_H-modules. From these observations we obtain:

The set
$$\mathcal{G}^H = \{I \in \mathcal{L}(E_H) \mid J_H \subseteq I\} = \{I \in \mathcal{L}(E_H) \mid MI = M\}$$

is a left Gabriel topology on E_H and we have the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Pres}^{G/H}[M] & \overset{\mathrm{HOM}_{G/K,R}(M,-)}{\underset{M \otimes_E -}{\rightleftarrows}} & (G/H, E, \mathcal{G}^{G/H})\text{-Gr} \\
\Big\| & & E \otimes_{E_H} - \Big\Updownarrow (-)_H \\
\mathrm{Pres}^{G/H}[M] & \overset{\mathrm{Hom}_{(G/H,R)\text{-Gr}}(M,-)}{\underset{M \otimes_{E_H} -}{\rightleftarrows}} & (E_H, \mathcal{G}^H)\text{-Mod}
\end{array}
$$

where all the arrows are equivalences.

If R is also strongly graded, then we obtain the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Pres}^{G/H}[M] & \overset{\mathrm{HOM}_{G/K,R}(M,-)}{\underset{M \otimes_E -}{\rightleftarrows}} & (G/H, E, \mathcal{G}^{G/H})\text{-Gr} \\
R \otimes_{R_H} - \Big\Updownarrow (-)_H & & E \otimes_{E_H} - \Big\Updownarrow (-)_H \\
\mathrm{Pres}[M_H] & \overset{\mathrm{Hom}_{R_H}(M_H,-)}{\underset{M_H \otimes_{E_H} -}{\rightleftarrows}} & (E_H, \mathcal{G}^H)\text{-Mod}
\end{array}
$$

Similar statements are true if we replace Pres with $\mathcal{C}$ or GF.

4.4. Our results generalize [3, Theorems 4.8, 4.12 and 4.15], which were obtained in the case when M is the canonical generator $\bigoplus_{g \in G} R(g)$ of R-Gr (actually only the relationship between the categories R-Mod, R-Gr, E-Mod and E-Gr was considered there). Indeed, it is enough to observe that our ideal J coincide with the ideal $\tau(U)$ introduced in [3, p. 141].

Theorem 3.9 and Corollary 3.11 above should also be compared with the results of [15, Sections 2 and 3], but strictly speaking it does not generalize them. The point is that in the particular case of $M = \bigoplus_{g \in G} R(g)$, the ring E can be canonically identified with a ring of matrices with elements in R, and it contains a G-graded subring $R\{G\}$, which coincides with E if and only if the group G is finite. This subring still retains the essential information about E, and can be obtain equivalences which are similar to Theorem 3.12. This approach was generalized in a different direction by G. Abrams and C. Menini [1].

4.5. Assume that the ideal J is pure (see 1.4). We know that the localizing subcategory corresponding to the G/H-graded topology $\mathcal{G}^{G/H}$ is

$$\mathcal{K}^{G/H} = \{X \in (G/H, E)\text{-Gr} \mid JX = 0\}.$$

Denote also

$$\mathcal{J}^{G/H} = \{X \in (G/H, E)\text{-Gr} \mid JX = X\}.$$

Using an appropriate graded version of [2, Theorem 1.6] (which can be easily deduced) we obtain that $\mathcal{J}^{G/H}$ is a localizing subcategory of $(G/H, E)$-Gr and there are category equivalences

$$(G/H, E)\text{-Gr}/\mathcal{J}^{G/H} \cong (G/H, E/J)\text{-Gr} \cong \mathcal{K}^{G/H}$$

and

$$(G/H, E, \mathcal{G}^{G/H})\text{-Gr} \cong (G/H, E)\text{-Gr}/\mathcal{K}^{G/H} \cong \mathcal{J}^{G/H}.$$

If moreover one can write $J = \sum_{\lambda \in \Lambda} p_\lambda E$, where $\{p_\lambda \mid \lambda \in \Lambda\}$ is a nonempty set of orthogonal idempotents in E_1, then $\mathcal{J}^{G/H}$ is equivalent to the category $(G/H, J)$-Gr of G/H-graded unital J-module.

If in addition E is strongly graded, then $(G/H, E, \mathcal{G}^{G/H})$-Gr is equivalent to J_H-Mod. Note that all the assumptions made here are satisfied if $M = \bigoplus_{g \in G} R(g)$ is the canonical generator of R-Gr.

Acknowledgements

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